INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

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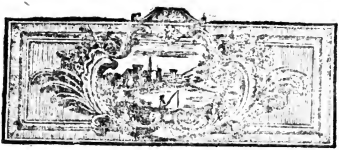
EDITIO NOVA.

TOMUS PRIMUS.



LUGDUNI,
Apud BERNUSET, DELAMOLLIERE, FALQUE & Soc.

1797.



ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XL VIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

NEC NON

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME,

PATRONOS Euleriano *ſcripto quærere neceſſe neutiquam eſſe, Mathematicarum Diſciplinarum cultoribus ſatis conſtat. Sciunt utique illi, varias earum partes novis eum luminibus ſic illuſtraſſe, ut indè meritò clariſſimi rerum in his abſtruſiſſimarum interpretis locum ſit conſequutus. Quem quin egregiè tueatur, immò tollat ſe altiùs quoque opere iſthoc, nemo dubitabit, certior hiſce factus, indulſiſſe* Te *mihi, ut illuſtriſſimo nomini* Tuo *dicatum publicè prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipſe, ut eo uterer, lubens conceſſit; & cum in rem meam faciat omnimodò, quî neglexiſſem?*

Ab his equidem, quibus Libros inscribunt, sibi nescio quid ideò deberi, plerique tacitè constituunt; acceptaque beneficia quodammodo remunerari, ut sese ferè nexu liberent omni. Ego verò secus sentio. Mihi certè merum est beneficium patroni, quòd scriptoris aut excusoris opera id genus honore condecorari patiantur. Hac mente utique Tibi, Vir Illustrissime, *animi gratissimi summæque observantiæ professionem hisce publicam excipias, rogo.*

Paratum promptumque semper juvandis litterarum studiis qui Te *novit, & notus vel hoc nomine es cuicumque in Republica doctorum Europæ totius non hospiti, plurimis officiis meæ etiam conditionis homines à* Te *affectos fuisse statuat necesse est. Nempe, tanquam* Tibi *uni esset injunctum curare, ut floreant humanum ingenium illustrantes scientiæ omnes, hominumque in usus adinventæ artes, ad singulis inservientium artifices etiam* Te *demittere dignaris, vel ab illa sublimium rerum perscrutatione, Cælive ipsius* Tibi *tam nota regione, ut quæ hucusque mentes hominum metu complebant phænomena minus intellecta, per* Te *jam grato tantum admirationis sensu contemplentur, earumque causas habeant perspectas.*

Hinc ille veluti ex condicto Academiarum Orbis eruditi concursus, ut adlectum Te *cœtui suo consequerentur, ornamento aliàs carituro insigni, quo cæteras nollent præ se frui. Hinc inprimis Illustrissimæ Parisiensis de* Te *judicium, cum ageretur*

de successore sufficiendo in locum emeriti Fontenellii , *Viri ; cujus ex ore calamoque fluere Scientiarum Artiumque omnium exquisitiores divitiæ , elegantiæque universæ perpetuo visæ sunt , & videbuntur dum sani sensus quicquam humano ingenio erit.* Tibi , *scilicet , Commentariorum Academiæ conscribendorum provincia , cui præfectus ille erat , demandabatur continuo ; quam , ut ornare diutiùs voluisses , docti omnes optabant : hoc uno minùs dolentes* Te *aliter censuisse , quod aliis* Tibi *magis placituris , profuturisque nihilominus litteris in universum eruditionis ingeniive thesauros impenderes. Quod ut ad ultimas usque metas hominum vitæ positas incolumis , florens , atque beatus præstes , omni votorum contentione precor. Vale !*

Datam *Lausannæ* die 1. Aprilis,
Anni Æræ Dionys. 1743.

PRÆFATIO.

Sæpe numero animadverti, maximam difficultatum partem, quas Matheſeos cultores in addiſcenda Analyſi infinitorum offendere ſolent, inde oriri, quod, Algebra communi vix apprehenſa, animum ad illam ſublimiorem artem appellant; quo fit, ut non ſolum quaſi in limine ſubſiſtant, ſed etiam perverſas ideas illius infiniti, cujus notio in ſubſidium vocatur, ſibi forment. Quanquam autem Analyſis infinitorum non perfectam Algebræ communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæſtiones, quarum evolutio diſcentium animos ad ſublimiorem ſcientiam præparare valet, quæ tamen in communibus Algebræ elementis, vel omittuntur, vel non ſatis accurate pertractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congeſſi, hunc defectum abunde ſupplere queant. Non ſolum enim operam dedi, ut eas res, quas Analyſis infinitorum abſolute requirit, uberius atque diſtinctius exponerem, quam vulgo fieri ſolet; ſed etiam ſatis multas quæſtiones enodavi, quibus Lectores ſenſim & quaſi præter expectationem ideam infiniti ſibi familiarem reddent. Plures quoque quæſtiones per præcepta communis Algebræ hic reſolvi, quæ vulgo in Analyſi infini-

torum tractantur: quo facilius deinceps utriusque Methodi summus consensus eluceat.

Divisi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analysin pertinent, sum complexus: in posteriori vero, quæ ex Geometria sunt scitu necessaria, explicavi, quoniam Analysis infinitorum ita quoque tradi solet, ut simul ejus applicatio ad Geometriam ostendatur. In utroque autem prima Elementa prætermisi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diversis principiis petita reperiuntur.

In primo igitur Libro, cum universa Analysis infinitorum circa quantitates variabiles earumque Functiones versetur, hoc argumentum de Functionibus imprimis fusius exposui; atque Functionum tam transformationem, quam resolutionem & evolutionem per series infinitas demonstravi. Complures enumeravi Functionum species, quarum in Analysi sublimiori præcipue ratio est habenda. Primum eas distinxi in algebraicas & transcendentes; quarum illæ per operationes in Algebra communi usitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iisdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria subdivisio fit in rationales & irrationales, priores docui cum in partes simpliciores, tum in factores resolvere; quæ operatio in Calculo integrali maximum adjumentum affert; posteriores vero, quemadmodum idoneis substitutionibus ad formam rationalem perduci queant ostendi. Evolutio autem per series infinitas ad utrumque genus æque pertinet, atque etiam ad Functiones transcendentes summa cum utilitate applicari solet; atquantopere doctrina de seriebus infinitis Analysin sublimiorem amplificaverit, nemo est qui ignoret. Nonnulla igitur adjunxi Capita, quibus plurium serierum infinitarum proprietates, atque summas sum scrutatus; quarum quædam ita sunt comparatæ, ut sine subsidio Analysis infinitorum vix investigari posse videantur. Hujusmodi series

ſeries ſunt, quarum ſummæ exprimuntur, vel per Logarithmos vel Arcus circulares: quæ quantitates cum ſint tranſcendentes, dum per quadraturam Hyperbolæ & Circuli exhibentur, maximam partem demum in Analyſi infinitorum tractari ſunt ſolitæ. Poſtquam autem a poteſtatibus ad quantitates exponentiales eſſem progreſſus, quæ nil aliud ſunt niſi poteſtates, quarum exponentes ſunt variabiles; ex earum converſione maxime naturalem ac fœcundam Logarithmorum ideam ſum adeptus: unde non ſolum ampliſſimus eorum uſus ſponte eſt conſecutus, ſed etiam ex ea cunctas ſeries infinitas, quibus vulgo iſtæ quantitates repræſentari ſolent, elicere licuit: hincque adeo facillimus ſe prodidit modus Tabulas Logarithmorum conſtruendi. Simili modo in contemplatione Arcuum circularium ſum verſatus; quod quantitatum genus, etſi a Logarithmis maxime eſt diverſum, tamen tam arcto vinculo eſt connexum, ut dum alterum imaginarium fieri videtur, in alterum tranſeat. Repetitis autem ex Geometria quæ de inventione Sinuum & Coſinuum Arcuum multiplorum ac ſubmultiplorum traduntur, ex Sinu vel Coſinu cujuſque Arcus expreſſi Sinum Coſinumque Arcus minimi & quaſi evaneſcentis, quo ipſo ad ſeries infinitas ſum deductus: unde, cum Arcus evaneſcens Sinui ſuo ſit æqualis, Coſinus vero radio, quemvis Arcum cum ſuo Sinu & Coſinu ope ſerierum infinitarum comparavi. Tum vero tam varias expreſſiones cum finitas tum infinitas pro hujus generis quantitatibus obtinui, ut ad earum naturam perſpiciendam Calculo infiniteſimali prorſus non amplius eſſet opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in univerſa Analyſi ſummus extat uſus, ita quantitates circulares ad certam quoque Algorithmi normam perduxi; ut in calculo æque commode ac Logarithmi & ipſæ quantitates algebraicæ tractari poſſent. Quantum autem hinc utilitatis ad reſolutionem difficillimarum quæſtionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analyſi infinitorum plurima ſpeci-

mina proferri possent, nisi jam satis essent cognita & in dies magis multiplicarentur. Maximum autem hæc investigatio attulit adjumentum ad Functiones fractas in factores reales resolvendas; quod argumentum, cum in Calculo integrali sit prorsus necessarium, diligentius enucleavi. Series postmodum infinitas, quæ ex hujusmodi Functionum evolutione nascuntur & quæ recurrentium nomine innotuerunt, examini subjeci; ubi earum tam summas quam terminos generales, aliasque insignes proprietates exhibui: & quoniam ad hæc resolutio in factores manuduxit, ita vicissim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conflata per multiplicationem in series explicentur, perpendi. Quod negotium non solum ad cognitionem innumerabilium serierum viam aperuit, sed quia hoc modo series in producta ex infinitis factoribus constantia resolvere licebat, satis commodas inveni expressiones numericas, quarum ope Logarithmi Sinuum, Cosinuum & Tangentium facillime supputari possunt. Præterea quoque ex eodem fonte solutiones plurium quæstionum, quæ circa partitionem numerorum proponi possunt, derivavi; cujusmodi quæstiones sine hoc subsidio vires Analyseos superare videantur. Hæc tanta materiarum diversitas in plura volumina facile excrescere potuisset; sed omnia, quantum fieri potuit, tam succincte proposui, ut ubique fundamentum clarissime quidem explicaretur, uberior vero amplificatio industriæ Lectorum relinqueretur; quo habeant, quibus vires suas exerceant, finesque Analyseos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non solum multa plane nova contineri; sed etiam fontes esse detectos, unde plurima insignia inventa adhuc hauriri queant. Eodem instituto sum usus in altero Libro, ubi, quæ vulgo ad Geometriam sublimiorem referri solent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere solæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita proposui, ut ad scrutationem naturæ

quarumvis Linearum Curvarum cum utilitate adhiberi posset. Ad hoc nullum aliud subsidium affero, præter æquationem, qua cujusque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potissimum in Sectionibus Conicis præstitisse mihi sum visus; quæ antehac vel secundum solam Geometriam vel per Analysin quidem, sed nimis imperfecte ac minus naturaliter, tractari sunt solitæ. Ex æquatione scilicet generali pro Lineis secundi ordinis primum earum proprietates generales explicavi, tum eas in genera seu species subdivisi; respiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito spatio includatur. Priori autem casu insuper dispiciendum erat, quot sint rami in infinitum excurrentes, & cujus naturæ sint singuli; an habeant Lineas rectas asymtotas, an minus. Sicque obtinui tres consuetas Sectionum Conicarum species; quarum prima est Ellipsis, tota in spatio finito contenta; secunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas asymptotas convergentes; tertia vero species prodiit Parabola duos habens ramos infinitos asymtotis destitutos. Simili porro ratione Lineas tertii ordinis sum persecutus, quas, post expositas earum proprietates generales, divisi in sedecim genera; ad eaque omnes septuagenta duas species NEWTONI revocavi. Ipsam vero methodum ita clare descripsi, ut pro quovis Linearum ordine sequente divisio in genera facillime institui queat; cujus negotii periculum quoque feci in Lineis quarti ordinis. His deinde, quæ ad ordines Linearum pertinent, expeditis, reversus sum ad generales omnium Linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales, atque etiam ipsam curvaturam, quæ per radium osculi æstimari solet: quæ etsi nunc quidem plerumque Calculo differentiali absolvuntur, tamen idem per solam communem Algebram hic

præſtiti, ut deinceps tranſitus ab Analyſi finitorum ad Analyſin infinitorum eo facilior reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuſpides, puncta duplicia, ac multiplicia; modumque expoſui hæc omnia ex æquationibus ſine ulla difficultate definiendi. Interim tamen non nego, has quæſtiones multo facilius Calculi differentialis ope enodari poſſe. Attigi quoque controverſiam de cuſpide ſecundi ordinis, ubi ambo arcus in cuſpidem coeuntes curvaturam in eandem partem vertunt; eamque ita compoſuiſſe mihi videor, ut nullum dubium amplius ſupereſſe poſſit. Denique adjunxi aliquot Capita, in quibus Lineas Curvas, quæ datis proprietatibus gaudeant, invenire docui; pleraque tandem Problemata circa ſingulares Circuli ſectiones ſoluta dedi. Quæ cum ſint ea ex Geometria, quæ ad Analyſin infinitorum addiſcendam maximum adminiculum afferre videntur, Appendicis loco ex Stereometria Theoriam ſolidorum eorumque ſuperficierum per Calculum propoſui & quemadmodum cujuſque ſuperficiei natura per æquationem inter tres variabiles exponi queat, oſtendi. Hinc, ſuperficiebus inſtar linearum in ordines digeſtis, ſecundum dimenſionum quas variabiles in æquatione conſtituunt numerum, in primo ordine ſolam ſuperficiem planam contineri oſtendi. Superficies vero ſecundi ordinis, ratione habita partium in infinitum expanſarum, in ſex genera diviſi; ſimilique modo pro ceteris ordinibus diviſio inſtitui poterit. Contemplatus ſum quoque interſectiones duarum ſuperficierum; quæ cum plerumque ſint curvæ non in eodem plano ſitæ, quemadmodum æquationibus comprehendi queant, monſtravi. Tandem etiam poſitionem planorum tangentium, atque rectarum, quæ ad ſuperficies ſint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem

genere elaborarunt, fecerim. Cum enim mihi propositum esset omnia quam brevissime pertractare, Historia cujusque Problematis magnitudinem operis non mediocriter auxisset. Interim tamen pleræque quæstiones, quæ alibi quoque solutæ reperiuntur, hic solutiones ex aliis principiis sunt nactæ; ita ut non exiguam partem mihi vindicare possem. Spero autem cum ista, tum ea potissimum, quæ prorsus nova hic proferuntur, plerisque, qui hoc studio delectantur, non ingrata esse futura.

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APPENDIX.

INTRO-

INTRODUCTIO
IN
ANALYSIN INFINITORUM.
LIBER PRIMUS,

CONTINENS

Explicationem de Functionibus quantitatum variabilium ; earum resolutione in Factores , atque evolutione per Series infinitas : una cum doctrina de Logarithmis , Arcubus circularibus , eorumque Sinubus & Tangentibus ; pluribusque aliis rebus , quibus Analysis infinitorum non mediocriter adjuvatur.

LIBER PRIMUS.

CAPUT PRIMUM.

DE FUNCTIONIBUS IN GENERE.

QUANTITAS *conſtans eſt quantitas determinata, perpetuo eumdem valorem ſervans.*

Ejuſmodi quantitates ſunt numeri cujuſvis generis, quippe qui eumdem, quem ſemel obtinuerunt, valorem conſtanter conſervant : atque ſi hujuſmodi quantitates conſtantes per characteres indicare convenit, adhibentur litteræ Alphabethi initiales a, b, c, &c. In Analyſi quidem communi, ubi tantum quantitates determinatæ conſiderantur, hæ litteræ Alphabethi priores quantitates cognitas denotare ſolent, poſteriores vero quantitates incognitas; at in Analyſi ſublimiori hoc diſcrimen non tantopere ſpectatur, cum hic ad illud quantitatum diſcrimen præcipue reſpiciatur, quo aliæ conſtantes, aliæ vero variabiles ſtatuuntur.

LIB. I. 2. *Quantitas variabilis est quantitas indeterminata seu universalis, quæ omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur ideæ specierum & generum; ita quantitas variabilis est genus, sub quo omnes quantitates determinatæ continentur. Hujusmodi autem quantitates variabiles per litteras Alphabethi postremas z, y, x, &c. repræsentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit Functio ipsius z: Sic $a + 3z$; $az - 4zz$; $az + b\sqrt{(aa - zz)}$; c^z; &c. sunt Functiones ipsius z.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc Functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etsi hæc Functio $\sqrt{(9 - zz)}$, numeris realibus loco z substituendis, nunquam valorem terna-

rio majorem recipere poteſt; tamen ipſi z valores imaginarios tribuendo ut $5\sqrt{-1}$, nullus aſſignari poterit valor determinatus quin ex formula $\sqrt{(9-zz)}$ elici queat. Occurrunt autem nonnunquam Functiones tantum apparentes, quæ, utcunque quantitas variabilis varietur, tamen uſque eumdem valorem retinent, ut z^0; 1^z; $\frac{aa-az}{a-z}$, quæ, etſi ſpeciem Functionis mentiuntur, tamen revera ſunt quantitates conſtantes. CAP. I.

6. Præcipuum Functionum diſcrimen in modo compoſitionis, quo ex quantitate variabili & quantitatibus conſtantibus formantur, poſitum eſt.

Pendet ergo ab Operationibus quibus quantitates inter ſe componi & permiſceri poſſunt: quæ Operationes ſunt Additio & Subtractio; Multiplicatio & Diviſio · Evectio ad Poteſtates & Radicum Extractio; quo etiam Reſolutio Æquationum eſt referenda. Præter has Operationes, quæ algebraïcæ vocari ſolent, dantur complures aliæ tranſcendentes, ut Exponentiales, Logarithmicæ, atque innumerabiles aliæ, quas Calculus integralis ſuppeditat.

Interim ſpecies quædam Functionum notari poſſunt; ut multipla $2z$; $3z$; $\frac{1}{2}z$; az; &c. & Poteſtates ipſius z, ut z^2; z^3; $z^{\frac{1}{2}}$; z^{-1}; &c. quæ, uti ex unica operatione ſunt deſumtæ, ita expreſſiones quæ ex operationibus quibuſcunque naſcuntur, Functionum nomine inſigniuntur.

7. Functiones dividuntur in Algebraïcas & Tranſcendentes: illæ ſunt, quæ componuntur per operationes algebraïcas ſolas; hæ vero, in quibus operationes tranſcendentes inſunt.

Sunt ergo multiplæ ac Poteſtates ipſius z Functiones algebraïcæ; atque omnes omnino expreſſiones, quæ per operationes algebraïcas ante memoratas formantur, cujuſmodi eſt $\frac{a+bz^n-c\sqrt{(2z-zz)}}{aaz-3bz}$. Quinetiam Functiones algebraïcæ ſæpenumero nequidem explicite exhiberi poſſunt, cujuſmodi Functio ipſius z eſt Z, ſi definiatur per hujuſmodi æquationem; $Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Quanquam enim hæc

 æquatio resolvi nequit; tamen constat Z æquari expressioni cuipiam ex variabili z & constantibus compositæ; ac propterea fore Z Functionem quamdam ipsius z. Cæterum de Functionibus transcendentibus notandum est, eas demum fore transcendentes, si operatio transcendens non solum ingrediatur, sed etiam quantitatem variabilem afficiat. Si enim operationes transcendentes tantum ad quantitates constantes pertineant, Functio nihilominus algebraïca est censenda: uti si c denotet circumferentiam Circuli, cujus radius sit $= 1$, erit utique c quantitas transcendens, verumtamen hæ expressiones $c + z$; cz^2; $4z^c$ &c. erunt Functiones algebraïcæ ipsius z. Parvi quidem est momenti dubium quod a quibusdam movetur, utrum ejusmodi expressiones z^c Functionibus algebraïcis annumerari jure possint, necne; quinetiam Potestates ipsius z, quarum exponentes sint numeri irrationales, uti $z^{\sqrt{2}}$, nonnulli maluerunt Functiones interscendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ subdividuntur in Rationales & Irrationales: illæ sunt, si quantitas variabilis in nulla irrationalitate involvitur; hæ vero, in quibus signa radicalia quantitatem variabilem afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Additionem, Subtractionem, Multiplicationem, Divisionem, & Evectionem ad Potestates, quarum exponentes sint numeri integri, non insunt: erunt adeo $a + z$; $a - z$; az; $\frac{aa + zz}{a + z}$; $az^3 - bz^4$; &c. Functiones rationales ipsius z. At hujusmodi expressiones $\sqrt{z}$; $a + \sqrt{(aa - zz)}$; $\sqrt[3]{(a - 2z + zz)}$: $\frac{aa - z\sqrt{(aa + zz)}}{a + z}$ erunt Functiones irrationales ipsius z.

Hæ commode distinguntur in Explicitas & Implicitas.

Explicitæ sunt, quæ per signa radicalia sunt evolutæ, cujusmodi exempla modo sunt data. Implicitæ vero Functiones irrationales sunt quæ ex resolutione æquationum ortum habent. Sic Z erit Functio irrationalis implicita ipsius z, si per hujusmodi æquationem $Z^5 = azZ^2 - bz^5$ definiatur; quoniam valorem

explicitum pro Z, admissis etiam signis radicalibus, exhibere non licet; propterea quod Algebra communis nondum ad hunc perfectionis gradum est evecta.

9. *Functiones rationales denuo subdividuntur in Integras & Fractas.*

In illis neque z usquam habet exponentes negativos, neque expressiones continent fractiones, in quarum denominatores quantitas variabilis z ingrediatur: unde intelligitur Functiones fractas esse, in quibus denominatores z continentes, vel exponentes negativi ipsius z occurrant. Functionum integrarum hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4 + fz^5 +$ &c. nulla enim Functio ipsius z integra excogitari potest, quæ non in hac expressione contineatur. Functiones autem fractæ omnes, quia plures fractiones in unam cogi possunt, continebuntur in hac Formula:

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + fz^5 + \&c.}{\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \zeta z^5 + \&c.}$$

ubi notandum est quantitates constantes a, b, c, d, &c. α, β, γ, δ, &c. sive sint affirmativæ, sive negativæ, sive integræ sive fractæ, rationales sive irrationales, sive etiam transcendentes, naturam Functionum non mutare.

10. *Deinde potissimum tenenda est Functionum divisio in Uniformes ac Multiformes.*

Functio autem uniformis est, quæ si quantitati variabili z valor determinatus quicunque tribuatur, ipsa quoque unicum valorem determinatum obtineat. Functio autem Multiformis est, quæ, pro unoquoque valore determinato in locum variabilis z substituto, plures valores determinatos exhibet. Sunt igitur omnes Functiones rationales, sive integræ sive fractæ, Functiones uniformes; quoniam ejusmodi expressiones, quicunque valor quantitati variabili tribuatur, non nisi unicum valorem præbent. Functiones autem irrationales omnes sunt multiformes; propterea quod signa radicalia sunt ambigua, & geminum valorem involvunt. Dantur autem quoque inter Functiones transcendentes, & uniformes, & multiformes: quinetiam habentur Func-

 tiones infinitiformes; cujusmodi est Arcus Circuli Sinui z respondens; dantur enim Arcus circulares innumerabiles qui omnes eumdem habeant Sinum. Denotent autem hæ litteræ P, Q, R, S, T &c. singulæ Functiones uniformes ipsius z.

11. *Functio biformis ipsius z est ejusmodi Functio, quæ pro quovis ipsius z valore determinato, geminum valorem præbeat.*

Hujusmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z + zz)}$: quicunque enim valor pro z statuatur expressio $\sqrt{(2z + zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero Z erit Functio biformis ipsius z, si determinetur per æquationem quadraticam $Z^2 - PZ + Q = 0$: si quidem P & Q fuerint Functiones uniformes ipsius z. Erit namque $Z = \frac{1}{2}P \pm \sqrt{(\frac{1}{4}P^2 - Q)}$; ex quo patet cuique valori determinato ipsius z duplicem valorem determinatum ipsius Z respondere. Hic autem notandum est, vel utrumque valorem Functionis Z esse realem, vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura æquationum, binorum valorum ipsius Z summa $= P$, ac productum $= Q$.

12. *Functio triformis ipsius z est, quæ pro quovis ipsius z valore, tres valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum cubicarum originem trahunt. Si enim fuerint P, Q, & R Functiones uniformes, sitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z Functio triformis ipsius z; quia pro quolibet valore determinato ipsius z triplicem valorem obtinet. Tres isti ipsius Z valores unicuique valori ipsius z respondentes, vel erunt omnes reales, vel unicus erit realis, dum bini reliqui sunt imaginarii. Cæterum constat horum trium valorum summam perpetuo esse $= P$; summam factorum ex binis esse $= Q$, & productum ex omnibus tribus esse $= R$.

13. *Functio quadriformis ipsius z est, quæ pro quovis ipsius z valore quatuor valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum biquadraticarum

ticarum nascuntur. Quod si enim P, Q, R, & S denotent Functiones uniformes ipsius z, fueritque $Z^4 - PZ^3 + QZ^2 - RZ + S = 0$, erit Z Functio quadriformis ipsius z; eo quod cuique valori ipsius z quadruplex valor ipsius Z respondet. Quatuor horum valorum ergo, vel omnes erunt reales, vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo summa horum quatuor valorum ipsius Z est $= P$, summa factorum ex binis $= Q$, summa factorum ex ternis $= R$, ac productum omnium $= S$. Similia autem modo comparata est ratio Functionum quinqueformium & sequentium.

14. *Erit ergo* Z *Functio multiformis ipsius* z, *quæ, pro quovis valore ipsius* z, *tot exhibet valores quot numerus* n *continet unitates; si* Z *definiatur per hanc æquationem* $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0$.

Ubi quidem notandum est n esse oportere numerum integrum; atque perpetuo, ut dijudicari possit quam multiformis sit Functio Z ipsius z, æquatio, per quam Z definitur, reduci debet ad rationalitatem; quo facto exponens maximæ potestatis ipsius Z indicabit quæsitum valorum numerum cuique ipsius z valori respondentium. Deinde quoque tenendum est litteras P, Q, R, S, &c. denotare debere Functiones uniformes ipsius z: si enim aliqua earum jam esset Functio multiformis, tum Functio Z multo plures præbitura esset valores unicuique valori ipsius z respondentes, quam quidem numerus dimensionum ipsius Z indicaret. Semper autem, si qui valores ipsius Z fuerint imaginarii, eorum numerus erit par; unde intelligitur, si fuerit n numerus impar, perpetuo unum ad minimum valorem ipsius Z fore realem: contra autem fieri posse, si numerus n fuerit par, ut nullus prorsus valor ipsius Z sit realis.

15. *Si* Z *ejusmodi fuerit Functio multiformis ipsius* z *ut perpetuo nonnisi unicum valorem exhibeat realem; tum* Z *Functionem uniformem ipsius* z *mentietur, ac plerumque loco Functionis uniformis usurpari poterit.*

Ejusmodi Functiones erunt $\sqrt[3]{P}$, $\sqrt[5]{P}$, $\sqrt[7]{P}$, &c. quippe quæ perpetuo nonnisi unicum valorem realem præbent, reliquis omnibus existentibus imaginariis, dummodo P fuerit Functio uniformis ipsius z. Hanc ob rem hujusmodi expressio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, Functionibus uniformibus annumerari poterit; sive m fuerit numerus par sive impar. Quod si autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem, vel duos; ex quo ejusmodi expressiones $P^{\frac{m}{n}}$, existente n numero pari, eodem jure Functionibus biformibus accenseri poterunt: siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit* y *Functio quæcunque ipsius* z *; tum vicissim* z *erit Functio ipsius* y.

Cum enim y sit Functio ipsius z, sive uniformis sive multiformis; dabitur æquatio, qua y per z & constantes quantitates definitur. Ex eadem vero æquatione vicissim z per y & constantes definiri poterit; unde quoniam y est quantitas variabilis, z æquabitur expressioni ex y & constantibus compositæ, eritque adeo Functio ipsius y. Hinc quoque patebit quam multifomis Functio futura sit z ipsius y: fierique potest ut, etiamsi y fuerit Functio uniformis ipsius z, tamen z futura sit Functio multiformis ipsius y. Sic si y ex hac æquatione per z definiatur; $y^3 = ayz - bzz$; erit utique y Functio triformis ipsius z, contra vero z Functio tantum biformis ipsius y.

17. *Si fuerint* y & x *Functiones ipsius* z, *erit quoque* y *Functio ipsius* x, & *vicissim* x *Functio ipsius* y.

Cum enim sit y Functio ipsius z, erit quoque z Functio ipsius y: similique modo erit etiam z Functio ipsius x. Hanc ob rem Functio ipsius y æqualis erit Functioni ipsius x; ex qua æquatione & y per x & vice versa x per y definiri poterit: quocirca manifestum est esse y Functionem ipsius x, atque x Functionem

ipſius y. Sæpiſſime quidem has Functiones explicite exhibere non licet ob defectum Algebræ; interim tamen nihilo minus, quaſi omnes æquationes reſolvi poſſent, hæc Functionum reciprocatio perſpicitur. Ceterum per methodum in Algebra traditam, ex datis binis æquationibus, quarum altera continet y & z, altera vero x & z, per eliminationem quantitatis z formabitur una æquatio relationem inter x & y exprimens. CAP. I.

18. *Species denique quædam Functionum peculiares ſunt notandæ; ſic Functio par ipſius* z *eſt, quæ eundem dat valorem, ſive pro* z *ponatur valor determinatus* $+$ k *ſive* $-$ k.

Hujuſmodi ergo Functio par ipſius z erit zz; ſive enim ponatur $z = +k$, ſive $z = -k$, eundem valorem præbebit expreſſio zz, nempe $zz = +kk$. Simili modo Functiones pares ipſius z erunt hæ ipſius z poteſtates z^4, z^6, z^8, & generatim omnis poteſtas z^m, ſi fuerit m numerus par, ſive affirmativus ſive negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur Functionem ipſius z uniformem, ſi n ſit numerus impar, perſpicuum eſt $z^{\frac{m}{n}}$ fore Functionem parem ipſius z, ſi m fuerit numerus par, n vero numerus impar. Hanc ob rem, expreſſiones ex hujuſmodi poteſtatibus utcunque compoſitæ præbebunt Functiones pares ipſius z; ſic Z erit Functio par ipſius z, ſi fuerit $Z = a + bz^2 + cz^4 + dz^6 +$ &c. item ſi fuerit $Z = \frac{a + bz^2 + cz^4 + dz^6 + \&c.}{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponentes fractos ipſius z introducendo, erit Z Functio par ipſius z ſi fuerit $Z = a + bz^{\frac{2}{3}} + cz^{\frac{4}{5}} + dz^{\frac{4}{3}}$ &c. vel $Z = a + bz^{-\frac{2}{3}} + cz^{-\frac{4}{3}} + dz^{-\frac{2}{5}} +$ &c. vel $Z = \frac{a + bz^{\frac{2}{7}} + cz^{-\frac{4}{3}} + dz^{\frac{8}{3}}}{\alpha + \beta z^{\frac{2}{3}} + \gamma z^{-\frac{4}{3}} + \delta z^{\frac{2}{7}}}$. Cujuſmodi expreſſiones, cum omnes ſint Functiones uniformes ipſius z, appellari poterunt Functiones pares uniformes ipſius z.

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19. *Functio multiformis par ipsius z est, quæ etiam si pro quovis valore ipsius z plures exhibeat valores determinatos, tamen eosdem valores præbet, sive ponatur $z = +k$, sive $z = -k$.*

Sit Z ejusmodi Functio multiformis par ipsius z; quoniam natura Functionis multifomis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimensiones, quot varios valores complectatur; manifestum est Z fore Functionem multiformem parem, si in æquatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit $Z^2 = a z^4 Z + b z^2$, erit Z Functio biformis par ipsius z; sin autem sit $Z^3 - a z^2 Z^2 + b z^4 Z - c z^8 = 0$, erit Z Functio triformis par ipsius z; atque generatim, si P, Q, R, S, &c. denotent Functiones uniformes pares ipsius z, erit Z Functio biformis par ipsius z si sit $Z^2 - P Z + Q = 0$. At Z erit Functio triformis par ipsius z si sit $Z^3 - P Z^2 + Q Z - R = 0$, & ita porro.

20. *Functio ergo, sive uniformis sive multiformis, par ipsius z erit ejusmodi expressio ex quantitate variabili z & constantibus conflata, in qua ubique numerus dimensionum ipsius z sit par.*

Hujusmodi ergo Functiones, præter uniformes quarum exempla ante sunt allata, erunt hæ expressiones $a + \sqrt{(bb - zz)}$; $a zz + \sqrt[3]{(a^6 z^4 - b z^2)}$ item $a z^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri posse, ut dicantur esse Functiones ipsius zz.

Sic enim ponatur $y = zz$, fueritque Z Functio quæcunque ipsius z; restituto ubique zz loco y, erit Z ejusmodi Functio ipsius z, in qua z ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius, Z occurunt $\sqrt{y}$: ac hujusmodi aliæ formæ, quæ, facto $y = zz$, signa radicalia amittunt. Quamvis enim sit $y + \sqrt{a y}$ Functio ipsius y, tamen posito $y = zz$, eadem expressio non erit Functio par ipsius z; cum fiat $y + \sqrt{a y} = zz + z\sqrt{a}$. Exclusis ergo his casibus, definitio ultima Functio-

num parium erit bona, atque ad ejusmodi Functiones formandas idonea.

21. *Functio impar ipsius z est ejusmodi Functio, cujus valor, si loco z ponatur — z, fit quoque negativus.*

Hujusmodi Functiones ergo impares erunt omnes potestates ipsius z, quarum exponentes sunt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, si ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expressio ex hujusmodi potestatibus composita erit Functio impar ipsius z; cujusmodi sunt, $az + bz^3 : az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{1}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perspicietur.

22. *Si Functio par ipsius z multiplicetur per z vel per ejusdem Functionem imparem quamcunque, productum erit Functio impar ipsius z.*

Sit P Functio par ipsius z, quæ idcirco manet eadem si loco z ponatur $-z$; quod si ergo in producto Pz, ponatur $-z$ loco z, prodibit $-Pz$; unde Pz erit Functio impar ipsius z. Sit jam P Functio par ipsius z, & Q Functio impar ipsius z; atque ex Definitione patet si loco z ponatur $-z$, valorem ipsius P manere eundem, at valorem ipsius Q abire in sui negativum $-Q$; quare productum PQ, posito $-z$ loco z, abibit in $-PQ$, hoc est in sui negativum; eritque ideo PQ Functio impar ipsius z. Sic cum sit $a + \sqrt{(aa + zz)}$ functio par, & z^3 Functio impar ipsius z, erit productum $az^3 + z^3\sqrt{(aa + zz)}$ Functio impar ipsius z; similique modo $z \times \frac{a + bzz}{\alpha + \gamma zz} = \frac{az + bz^3}{\alpha + \gamma zz}$ Functio impar ipsius z. Ex his vero etiam intelligitur, si duarum Functionum P & Q, quarum altera P est par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipsius z.

23. *Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod resultat erit Functio par.*

Sint Q & S Functiones impares ipsius z; ita ut, posito $-z$ loco z, Q abeat in $-Q$, & S in $-S$; atque perspicuum est tam productum QS, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur $-z$; ideoque esse utrumque Functionem parem ipsius z. Manifestum itaque porro est cujusque Functionis imparis quadratum esse Functionem parem; cubum vero Functionem imparem; biquadratum iterum Functionem parem, atque ita porro.

24. *Si fuerit* y *Functio impar ipsius* z; *erit vicissim* z *Functio impar ipsius* y.

Cum enim sit y Functio impar ipsius z; si ponatur $-z$ loco z, abibit y in $-y$. Quod si ergo z per y definiatur, necesse est ut posito $-y$ loco y, quoque z abeat in $-z$; eritque ideo z Functio impar ipsius y. Sic quia, posito $y = z^3$, est y Functio impar ipsius z; erit quoque, ex æquatione $z^3 = y$ seu $z = y^{\frac{1}{3}}$, z Functio impar ipsius y. Et quia si fuerit $y = az + bz^3$, est y Functio impar ipsius z, erit vicissim, ex æquatione $bz^3 + az = y$, valor ipsius z per y expressus Functio impar ipsius y.

25. *Si natura Functionis* y *per ejusmodi æquationem definiatur, in cujus singulis terminis numerus dimensionum, quas* y & z *occupant conjunctim, sit vel par ubique, vel impar; tum erit* y *Functio impar ipsius* z.

Quod si enim in ejusmodi æquatione ubique loco z scribatur $-z$; simulque $-y$ loco y; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero casu æquatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, si loco z ponatur $-z$, valor ipsius y abibit in $-y$, seu y erit Functio impar ipsius z. Sic si fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipsius z.

26. *Si Z fuerit Functio ipsius z, & Y Functio ipsius y, atque Y eodem modo definiatur per variabilem y & constantes, quo Z definitur per variabilem z & constantes; tum hæ Functiones Y et Z vocantur Functiones similes ipsarum y & z.*

Si scilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt Z & Y Functiones similes ipsarum z & y, similique modo in multiformibus, si fuerit $Z^3 = azz\,Z + b$ & $Y^3 = ayy\,Y + b$; erunt Z & Y Functiones similes ipsarum z & y. Hinc sequitur, si Y & Z fuerint hujusmodi Functiones similes ipsarum y & z, tum si loco z scribatur y, Functionem Z abituram esse in Functionem Y. Solet hæc similitudo etiam hoc modo verbis exprimi, ut Y talis Functio dicatur ipsius y, qualis Functio sit Z ipsius z. Hæ locutiones perinde occurrent, sive quantitates variabiles z & y a se invicem pendeant, sive secus: sic qualis Functio est $ay + by^2$ ipsius y, talis Functio erit $a(y+n) + b(y+n)^2$ ipsius $y + n$, existente scilicet $z = y + n$: tum qualis Functio est $\frac{a + bz + czz}{\alpha + \zeta z + \gamma zz}$ ipsius z, talis Functio erit $\frac{azz + bz + c}{\alpha zz + \zeta z + \gamma}$ ipsius $\frac{1}{z}$; posito $y = \frac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis Functionum, cujus per universam Analysin sublimiorem uberrimus est usus. Ceterum hæc in genere de natura Functionum unius variabilis sufficere possunt; cum plenior expositio in applicatione sequente tradatur.

CAPUT II.

De transformatione Functionum.

27. *FUNCTIONES in alias formas transmutantur, vel loco quantitatis variabilis aliam introducendo, vel eandem quantitatem variabilem retinendo.*

Quod si eadem quantitas variabilis servatur, Functio proprie mutari non potest. Sed omnis transformatio consistit in

LIB. I. alio modo eandem Functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Hujusmodi transformationes sunt, si loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aaz + 3azz + z^3$, vel $\frac{a}{a-z} + \frac{a}{a+z}$ loco $\frac{2aa}{aa-zz}$; vel $\sqrt{(1+zz)} + z$ loco $\frac{1}{\sqrt{(1+zz)} - z}$; quæ expressiones, etsi forma differunt, tamen revera congruunt. Sæpe numero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquæ, & hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quæ quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut Functio proposita succinctius & commodius exprimatur, uti si ista proposita fuerit ipsius z Functio, $a^4 - 4a^3z + 6aazz - 4az^3 + z^4$; si loco $a - z$ ponatur y, prodibit ista multo simplicior ipsius y Functio y^4: &, si habeatur hæc Functio irrationalis $\sqrt{(aa + zz)}$ ipsius z, si ponatur $z = \frac{aa - yy}{2y}$, ista Functio per y expressa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in sequens Caput differam, hoc Capite illum, qui sine substitutione procedit, expositurus.

29. *Functio integra ipsius z sæpenumero commode in suos factores resolvitur, sicque in productum transformatur.*

Quando Functio integra hoc pacto in factores resolvitur, ejus natura multo facilius perspicitur; casus enim statim innotescunt, quibus Functionis valor fit $= 0$. Sic hæc ipsius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo statim liquet Functionem propositam tribus casibus fieri $= 0$; scilicet si $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Istiusmodi factores, in quibus variabilis z nulla

nulla occurrit poteſtas quadrata vel altior, vocantur Factores ſimplices, ut diſtinguantur a Factoribus compoſitis, in quibus ipſius z ineſt quadratum vel cubus, vel alia poteſtas altior. Erit ergo in genere $f+gz$ forma Factorum ſimplicium, $f+gz+hzz$ forma Factorum duplicium, $f+gz+hzz+iz^3$ forma Factorum triplicium, & ita porro. Perſpicuum autem eſt Factorem duplicem duos complecti valores ſimplices, Factorem triplicem tres ſimplices, & ita porro. Hinc Functio ipſius z integra, in qua exponens ſummæ poteſtatis ipſius z eſt $= n$, continebit n Factores ſimplices; ex quo ſimul, ſi qui Factores fuerint vel duplices vel triplices, &c. numerus Factorum cognoſcetur.

29. *Factores ſimplices Functionis cujuſcunque integræ* Z *ipſius* z *reperiuntur, ſi Functio* Z *nihilo æqualis ponatur, atque ex hac æquatione omnes ipſius* z *radices inveſtigentur: ſingulæ enim ipſius* z *radices dabunt totidem Factores ſimplices Functionis* Z.

Quod ſi enim ex æquatione $Z=0$, fuerit quæpiam radix $z=f$, erit $z-f$ diviſor, ac proinde Factor Functionis Z, ſic igitur inveſtigandis omnibus radicibus æquationis $Z=0$, quæ ſint $z=f$, $z=g$, $z=h$; &c., Functio Z reſolvetur in ſuos Factores ſimplices, atque transformabitur in productum $Z=(z-f)(z-g)(z-h)$ &c.: ubi quidem notandum eſt ſi ſummæ poteſtatis ipſius z in Z non fuerit coefficiens $=+1$, tum productum $(z-f)(z-g)$ &c. inſuper per illum coefficientem multiplicari debere. Sic ſi fuerit $Z=Az^n+Bz^{n-1}+Cz^{n-2}+$ &c. erit $Z=A(z-f)(z-g)(z-h)$ &c. At ſi fuerit $Z=A+Bz+Cz^2+Dz^3+Ez^4+$ &c. atque æquationis $Z=0$ radices z repertæ ſint; $f; g; h; i;$ &c. erit $Z=A(1-\frac{z}{f})(1-\frac{z}{g})(1-\frac{z}{h})$ &c. Ex his autem viciſſim intelligitur, ſi Functionis Z Factor fuerit $z-f$, ſeu $1-\frac{z}{f}$; tum valorem Functionis in nihilum abire, ſi loco z ponatur f. Facto enim $z=f$, unus Factor $z-f$, ſeu $1-\frac{z}{f}$ Functionis Z, ideoque ipſa Functio Z evaneſcere debet.

30. *Factores simplices ergo erunt vel reales, vel imaginarii; &, si Functio Z habeat Factores imaginarios, eorum numerus semper erit par.*

Cum enim Factores simplices nascantur ex radicibus æquationis $Z = 0$, radices reales præbebunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum semper est par: quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel sex, &c. Quod si Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ producto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariorum $= \frac{Z}{P}$; hincque reale. Simili modo si Functio Z habeat quatuor, vel sex, vel octo, &c. Factores imaginarios; erit eorum productum semper reale: nempe æquale quoto, qui oritur, si Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit Q productum reale ex quatuor Factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejusmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ si negetur in duos Factores duplices reales resolvi posse, resolubilis erit statuenda in duos Factores duplices imaginarios, qui hujusmodi formam habebunt $zz - 2(p + q\sqrt{-1})z + r + s\sqrt{-1}$, & $zz - 2(p - q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non possunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus sequentes emergent quatuor Factores simplices imaginarii ipsius Q,

I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$
IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

Horum Factorum multiplicentur primus ac tertius in se invicem, posito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; eritque horum Factorum productum $= zz - (2p - \sqrt{2t + 2\sqrt{(tt+uu)}})z + pp + qq - p\sqrt{2t + 2\sqrt{(tt+uu)}} + q\sqrt{-2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}$; quod utique est reale. Simili autem modo productum ex Factoribus secundo & quarto erit reale nempe $= zz - (2p + \sqrt{2t + 2\sqrt{tt+uu}})z + pp + qq + p\sqrt{2t + 2\sqrt{(tt+uu)}} + q\sqrt{-2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}$. Quocirca productum propositum Q, quod in duos Factores duplices reales resolvi posse negabatur, nihilo minus actu in duos Factores duplices reales est resolutum.

32. *Si Functio integra* Z *ipsius* z *quotcunque habeat Factores simplices imaginarios, bini semper ita conjungi possunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum semper est par, sit is $= 2n$; ac primo quidem patet productum harum radicum imaginarium omnium esse reale. Quod si ergo duæ tantum radices imaginariæ habeantur, erit earum productum utique reale; sin autem quatuor habeantur Factores imaginarii, tum, uti vidimus, eorum productum resolvi potest in duos Factores duplices reales formæ $fzz + gz + h$. Quanquam autem eundem demonstrandi modum ad altiores potestates extendere non licet, tamen extra dubium videtur esse positum eandem proprietatem in quotcumque Factores imaginarios competere; ita ut semper loco $2n$ Factorum simplicium imaginariorum induci queant n Factores duplices reales. Hinc omnis Functio integra ipsius z resolvi poterit in Factores reales vel simplices vel duplices. Quod quamvis non summo rigore sit demonstratum, tamen ejus veritas in sequentibus magis corroborabitur, ubi hujus generis Functiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c. actu in istiusmodi Factores duplices reales resolventur.

LIB. I. 33. *Si Functio integra* Z , *posito* z = a , *induat valorem* A , *& posito* z = b , *induat valorem* B ; *tum , loco* z *valores medios inter* a *&* b *ponendo , Functio* Z *quosvis valores medios inter* A *&* B *accipere potest.*

Cum enim Z sit Functio uniformis ipsius z , quicunque valor realis ipsi z tribuatur , Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z, priore casu $z = a$, nanciscatur valorem A; posteriore casu $z = b$, autem , valorem B; ab A ad B transire non poterit , nisi per omnes valores medios transeundo. Quod si ergo æquatio $Z - A = 0$ habeat radicem realem , simulque $Z - B = 0$ radicem realem suppeditet ; tum æquatio quoque $Z - C = 0$ radicem habebit realem ; si quidem C intra valores A & B contineatur. Hinc si expressiones $Z - A$ & $Z - B$ habeant Factorem simplicem realem , tum expressio quæcunque $Z - C$ Factorem simplicem habebit realem , dummodo C intra valores A & B contineatur.

34. *Si in Functione integra* Z *exponens maximæ ipsius* z *potestatis fuerit numerus impar* 2 n + 1 , *tum ea Functio* Z *unicum ad minimum habebit Factorem simplicem realem.*

Habebit scilicet Z hujusmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} +$ &c. in qua si ponatur $z = \infty$, quia valores singulorum terminorum præ primo evanescunt , fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem simplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem simplicem realem ; sequitur etiam $Z + C$ habiturum esse Factorem simplicem realem , siquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc est si C fuerit numerus realis quicunque , sive affirmativus , sive negativus. Hanc ob rem , facto $C = 0$, habebit quoque ipsa Functio Z Factorem simplicem realem $z - c$; atque quantitas c

continebitur intra limites $+\infty$ & $-\infty$, eritque idcirco vel quantitas affirmativa, vel negativa, vel nihil. CAP. II.

35. *Functio igitur integra* Z, *in qua exponens maximæ potestatis ipsius* z *est numerus impar, vel unum habebit Factorem simplicem realem, vel tres, vel quinque, vel septem, &c.*

Cum enim demonstratum sit Functionem Z certo unum habere Factorem simplicem realem $z - c$; ponamus eam præterea unum Factorem habere $z - d$, atque dividatur Functio Z, in qua maxima ipsius z potestas sit z^{2n+1}, per $(z-c)(z-d)$, erit quoti maxima potestas $= z^{2n-1}$, cujus exponens, cum sit numerus impar, indicat denuo ipsius Z dari Factorem simplicem realem. Si ergo Z plures uno habeat Factores simplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel septem, &c. Erit scilicet numerus Factorum simplicium realium impar, & quia numerus omnium Factorum simplicium est $= 2n + 1$, erit numerus Factorum imaginariorum par.

36. *Functio integra* Z, *in qua exponens maximæ potestatis ipsius* z *est numerus par* 2n, *vel duos habebit Factores simplices reales, vel quatuor, vel sex, vel &c.*

Ponamus ipsius Z constare Factorum simplicium realium numerum imparem $2m + 1$; si ergo per horum omnium productum dividatur Functio Z, quoti maxima potestas erit $= z^{2n-2m-1}$, ejusque ideo exponens numerus impar; habebit ergo Functio Z præterea unum certo Factorem simplicem realem, ex quo numerus omnium Factorum simplicium realium ad minimum erit $= 2m + 2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integræ Factores simplices imaginarii sunt numero pares; quemadmodum quidem jam ante statuimus.

37. *Si in Functione integra* Z *exponens maximæ potestatis ipsius* z *fuerit numerus par, atque terminus absolutus, seu constans, signo — affectus, tum Functio* Z *ad minimum duos habet Factores simplices reales.*

LIB. I. Functio ergo Z, de qua hic sermo est, hujusmodi formam habebit $z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \ldots\ldots \pm \nu z - A$. Si jam ponatur $z = \infty$, fiet, uti supra vidimus, $Z = \infty$; atque, si ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem $z - 0$: unde cum 0 contineatur intra limites $-\infty$ & $+A$, sequitur $Z + 0$ habere Factorem simplicem realem $z - c$, ita ut c contineatur intra limites 0 & ∞. Deinde, cum posito $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habeat $z + \infty$, & $Z + A$ Factorem $z + 0$, sequitur quoque $Z + 0$ Factorem simplicem realem habere $z + d$; ita ut d intra limites 0 & ∞ contineatur; unde constat propositum. Ex his igitur perspicitur si Z talis fuerit Functio, qualis hic est descripta, æquationem $Z = 0$, duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic æquatio hæc $z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$, duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in Functione fracta, quantitas variabilis z tot vel plures habeat dimensiones in numeratore, quam in denominatore; tum ista Functio resolvi poterit in duas partes, quarum altera est Functio integra, altera fracta, in cujus numeratore quantitas variabilis z pauciores habeat dimensiones quam in denominatore.*

Si enim exponens maximæ potestatis ipsius z minor fuerit in denominatore quam in numeratore; tum numerator per denominatorem dividatur more solito, donec in quoto ad exponentes negativos ipsius z perveniatur; hoc ergo loco abrupta divisionis operatione quotus constabit ex parte integra atque fractione, in cujus numeratore minor erit dimensionum numerus ipsius z quam in denominatore; hic autem quotus Functioni propositæ est æqualis. Sic, si hæc proposita fuerit Functio fracta $\frac{1+z^4}{1+zz}$, ea per divisionem ita resolvetur:

$$zz + 1 \,)\, z^4 + 1 \,(\, zz - 1 + \frac{2}{1+zz}$$

$$\begin{array}{r} z^4 + zz \\ \hline -zz + 1 \\ -zz - 1 \\ \hline +2 \end{array}$$

eritque $\frac{1+z^4}{1+zz} = zz - 1 + \frac{2}{1+zz}$. Hujusmodi Functiones fractæ, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticæ vocari possunt fractiones spuriæ, vel Functiones fractæ spuriæ, quo distinguantur a Functionibus fractis genuinis, in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in Functionem integram, & Functionem fractam genuinam; hæcque resolutio per vulgarem divisionis operationem absolvetur.

39. *Si denominator Functionis fractæ duos habeat Factores inter se primos; tum ipsa Functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis Factoribus respective æquales.*

Quanquam hæc resolutio ad Functiones fractas spurias æque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accommodabimus. Resoluto autem denominatore hujusmodi Functionis fractæ in duos Factores inter se primos, ipsa Functio resolvetur in duas alias Functiones fractas genuinas, quarum denominatores sint illis binis Factoribus respective æquales; hæcque resolutio, si quidem fractiones sint genuinæ, unico modo fieri potest; cujus rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita hæc Functio fracta $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$, cujus denominator $1 + 4z^4$ cum sit æqualis huic producto $(1 + 2z + 2zz)(1 - 2z + 2zz)$, fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1 + 2z + 2zz$, alterius $1 - 2z + 2zz$: ad quas inveniendas, quia sunt genuinæ, statuantur numeratores illius $= \alpha + \beta z$, hujus $= \gamma + \delta z$, eritque per hypothesin

LIB. I. $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4} = \frac{\alpha + \beta z}{1 + 2z + 2zz} + \frac{\gamma + \delta z}{1 - 2z + 2zz}$: addantur actu hæ duæ fractiones, eritque summæ

Numerator	Denominator
$+\alpha - 2\alpha z + 2\alpha zz$ $+ \beta z - 2\beta zz + 2\beta z^3$ $+\gamma + 2\gamma z + 2\gamma zz$ $+ \delta z + 2\delta zz + 2\delta z^3$	$1 + 4z^4$

Cum ergo denominator æqualis sit denominatori fractionis propositæ, numeratores quoque æquales reddi debent: quod, ob tot litteras incognitas $\alpha, \beta, \gamma, \delta$, quot sunt termini æquales efficiendi, utique fieri, idque unico modo poterit: nanciscimur scilicet has quatuor æquationes

$$\text{I. } \alpha + \gamma = 1 \qquad \text{III. } 2\alpha - 2\beta + 2\gamma + 2\delta = 3$$
$$\text{II. } -2\alpha + \beta + 2\gamma + \delta = -2 \qquad \text{IV. } 2\beta + 2\delta = -4.$$

Hinc ob $\alpha + \gamma = 1$, & $\beta + \delta = -2$; æquationes II. & III. dabunt $\alpha - \gamma = 0$ & $\delta - \beta = \frac{1}{2}$; ex quibus fit $\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $\beta = \frac{-5}{4}$; $\delta = \frac{-3}{4}$; ideoque fractio proposita $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$, transformatur in has duas $\frac{\frac{1}{2} - \frac{5}{4}z}{1 + 2z + 2zz} + \frac{\frac{1}{2} - \frac{3}{4}z}{1 - 2z + 2zz}$. Simili autem modo facile perspicietur resolutionem semper succedere debere: quoniam semper tot litteræ incognitæ introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris Factores fuerint inter se primi.

40. *Functio igitur fracta* $\frac{M}{N}$ *in tot fractiones simplices formæ* $\frac{A}{p - qz}$ *resolvi poterit, quot Factores simplices habet denominator* N *inter se inæquales.*

Repræsentat

Repræfentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N fint Functiones integræ ipfius z, atque fumma poteftas ipfius z in M minor fit quam in N. Quod fi ergo denominator N in fuos Factores fimplices refolvatur, hique inter fe fuerint inæquales, expreffio $\frac{M}{N}$ in tot fractiones refolvetur, quot Factores fimplices in denominatore N continentur; propterea quod quifque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipfius N, is erit denominator fractionis cujufdam partialis, &, cum in numeratore hujus fractionis numerus dimenfionum ipfius z minor effe debeat quam in denominatore $p - qz$, numerator neceffario erit quantitas conftans. Hinc ex unoquoque Factore fimplici $p - qz$ denominatoris N nafcetur fractio fimplex $\frac{A}{p - qz}$; ita ut fumma omnium harum fractionum fit æqualis fractioni propofitæ $\frac{M}{N}$. CAP. II.

EXEMPLUM.

Sit, exempli caufa, propofita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores fimplices denominatoris funt z, $1 - z$, & $1 + z$, ifta Functio refolvetur in has tres fractiones fimplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores conftantes A, B, & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum fumma æquari debebit ipfi $1 + zz$, unde ifta æquatio oritur:

$$\begin{array}{l} A + Bz - Azz = 1 + zz = 1 + 0z + zz. \\ \quad + Cz + Bzz \\ \qquad\quad - Czz \end{array}$$

 quæ totidem comparationes præbet, quot ſunt litteræ incognitæ A, B, C; erit ſcilicet,

I°. $A = 1$.
II°. $B + C = 0$.
III°. $-A + B - C = 1$:

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C = -1$. Functio ergo propoſita $\frac{1+zz}{z-z^3}$ reſolvitur in hanc formam $\frac{1}{z} + \frac{1}{1-z} - \frac{1}{1+z}$. Simili autem modo intelligitur, quotcunque habuerit denominator N Factores ſimplices inter ſe inæquales, ſemper fractionem $\frac{M}{N}$ in totidem fractiones ſimplices reſolvi. Sin autem aliquot Factores fuerint æquales inter ſe, tum alio modo poſt-explicando reſolutio inſtitui debet.

41. *Cum igitur quilibet Factor ſimplex denominatoris* N *ſuppeditet fractionem ſimplicem pro reſolutione Functionis propoſitæ* $\frac{\mathrm{M}}{\mathrm{N}}$; *oſtendendum eſt quomodo ex Factore ſimplice denominatoris* N *cognito, fractio ſimplex reſpondens reperiatur.*

Sit $p - qz$ Factor ſimplex ipſius N, ita ut ſit $N = (p - qz)S$; atque S Functio integra ipſius z; ponatur fractio ex Factore $p - qz$ orta $= \frac{A}{p-qz}$, & ſit fractio ex altero Factore denominatoris S orianda $= \frac{P}{S}$, ita ut, ſecundum §. 39., futurum ſit $\frac{M}{N} = \frac{A}{p-qz} + \frac{P}{S} = \frac{M}{(p-qz)S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p-qz)S}$; quæ fractiones cum congruere debeant, neceſſe eſt ut $M - AS$ ſit diviſibile per $p - qz$; quoniam Functio integra P ipſi quoto æquatur. Quando vero $p - qz$ Diviſor exiſtit ipſius $M - AS$, hæc expreſſio poſito $z = \frac{p}{q}$ evaneſcit. Ponatur ergo ubique loco z hic valor conſtans $\frac{p}{q}$ in M.

& S, erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo modo reperitur numerator A fractionis quæsitæ $\frac{A}{p - qz}$; atque si ex singulis denominatoris N Factoribus simplicibus, dummodo sint inter se inæquales, hujusmodi fractiones simplices formentur, harum fractionum simplicium omnium summa erit æqualis Functioni propositæ $\frac{M}{N}$.

EXEMPLUM.

Sic, si in Exemplo præcedente $\frac{1 + zz}{z - z^3}$, ubi est $M = 1 + zz$, & $N = z - z^3$, sumatur z pro Factore simplice, erit $S = 1 - zz$, atque fractionis simplicis $\frac{A}{z}$ hinc ortæ erit numerator $A = \frac{1 + zz}{1 - zz} = 1$ posito $z = 0$, quem valorem z obtinet si ipse hic Factor simplex z nihilo æqualis ponatur. Simili modo si pro denominatoris Factore sumatur $1 - z$, ut fit $S = z + zz$ erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, & ex Factore $1 - z$ nascitur fractio $\frac{1}{1 - z}$. Tertius denique Factor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, posito $1 + z = 0$, seu $z = -1$, dabit $A = -1$, & fractionem simplicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3} = \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus formæ* $\frac{P}{(p - qz)^n}$, *cujus numerator* P *non tantam ipsius* z *potestatem involvit quantam denominator* $(p - qz)^n$, *transmutari potest in hujusmodi fractiones partiales*

LIB. I. $\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\ldots\ldots+\frac{K}{p-qz}$, *quarum omnium numeratores sint quantitates constantes.*

Quoniam maxima potestas ipsius z in P minor est quam z^n, erit z^{n-1}, ideoque P hujusmodi habebit formam:

$\alpha+\beta z+\gamma z^2+\delta z^3+\ldots\ldots\ldots+\varkappa z^{n-1}$

existente terminorum numero $=n$, cui æquari debet numerator summæ omnium fractionum partialium, postquam singulæ ad eundem denominatorem $(p-qz)^n$ fuerint perductæ: qui numerator propterea erit $=A+B(p-qz)+C(p-qz)^2+D(p-qz)^3+\ldots\ldots+K(p-qz)^{n-1}$. Hujus maxima ipsius z potestas est, ut ibi, z^{n-1}, atque tot habentur litteræ incognitæ $A, B, C, \ldots\ldots K$, (quarum numerus est $=n$,) quot sunt termini congruentes reddendi. Quamobrem litteræ constantes A, B, C, &c. ita definiri poterunt, ut fiat Functio fracta genuina $\frac{P}{(p-qz)^n}=\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\frac{D}{(p-qz)^{n-3}}+\ldots\ldots\ldots+\frac{K}{p-qz}$. Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p-qz)^2$, *sequenti modo fractiones partiales ex hoc Factore oriundæ reperientur.*

Cujusmodi fractiones partiales ex singulis Factoribus denominatoris simplicibus, qui alios sibi æquales non habeant, oriantur, ante est ostensum: nunc igitur ponamus duos Factores inter se esse æquales, seu, iis conjunctis, denominatoris N Factorem esse $(p-qz)^2$. Ex hoc ergo Factore per §. præced. duæ nascentur fractiones partiales hæ $\frac{A}{(p-qz)^2}+\frac{B}{p-qz}$. Sit au-

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2} + \frac{B}{p-qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones simplices junctim sumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S} = \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$, & $P = \frac{M - AS - B(p-qz)S}{(p-qz)^2}$ = Functioni integræ. Debet ergo $M - AS - B(p-qz)S$ divisibile esse per $(p-qz)^2$: sit primum divisibile per $p-qz$, atque tota expressio $M - AS - B(p-qz)S$ evanescet, posito $p - qz = 0$, seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z, eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; scilicet fractio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A constantem. Hoc invento quantitas $M - AS - B(p-qz)S$ etiam per $(p-qz)^2$ divisibilis esse debet, seu $\frac{M-AS}{p-qz} - BS$ denuo per $p-qz$ divisibile esse debet. Posito ergo ubique $z = \frac{p}{q}$ erit $\frac{M-AS}{p-qz} = BS$, ideoque $B = \frac{M-AS}{(p-qz)S} = \frac{1}{p-qz}\left(\frac{M}{S} - A\right)$, ubi notandum est, cum $M - AS$ divisibile sit per $p - qz$, hanc divisionem prius institui debere, quam loco z substituatur $\frac{p}{q}$. Vel ponatur $\frac{M-AS}{p-qz} = T$, eritque $B = \frac{T}{S}$ posito $z = \frac{p}{q}$; inventis ergo numeratoribus A & B, erunt fractiones partiales ex denominatoris N Factore $(p-qz)^2$ ortæ hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$.

Exemplum I.

Sit hæc proposita Functio fracta $\frac{1-zz}{zz(1+zz)}$ erit, ob de-

 nominatoris Factorem quadratum zz; $S = 1 + zz$ & $M = 1 - zz$. Sint fractiones partiales ex zz ortæ $\frac{A}{zz} + \frac{B}{z}$, erit $A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$, posito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = -2zz$ quod divisum per Factorem simplicem z, dabit $T = -2z$, hincque $B = \frac{T}{S} = \frac{-2z}{1 + zz}$, posito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\frac{1}{zz}$.

Exemplum II.

Sit hæc proposita Functio fracta $\frac{z^3}{(1-z)^2(1+z^4)}$, cujus, ob denominatoris Factorem quadratum $(1-z)^2$, fractiones partiales sint $\frac{A}{(1-z)^2} + \frac{B}{1-z}$. Erit ergo $M = z^3$ & $S = 1 + z^4$; ideoque $A = \frac{M}{S} = \frac{z^3}{1+z^4}$, posito $1 - z = 0$, seu $z = 1$: unde fit $A = \frac{1}{2}$. Prodibit ergo $M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = -\frac{1}{2} + z^3 - \frac{1}{2}z^4$, quod per $1 - z$ divisum dat $T = -\frac{1}{2} - \frac{1}{2}z - \frac{1}{2}zz + \frac{1}{2}z^3$; ideoque $B = \frac{T}{S} = \frac{-1 - z - zz + z^3}{2 + 2z^4}$, posito $z = 1$; ita ut fit $B = -\frac{1}{2}$; fractiones ergo partiales quæsitæ sunt $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^3$ *sequenti modo fractiones partiales ex hoc Factore oriundæ* $\frac{A}{(p-qz)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$ *reperientur.*

Ponatur $N = (p - qz)^3 S$, sitque fractio ex Factore S orta $= \frac{P}{S}$, erit $P = \frac{M - AS - B(p-qz)S - C(p-qz)^2 S}{(p-qz)^3}$ = Functioni integræ. Numerator ergo $M - AS - B(p-qz)S - C(p-qz)^2 S$ ante omnia divisibilis esse debet per $(p-qz)^3$; unde is, posito $p - qz = 0$, seu $z = \frac{p}{q}$, evanescere debet, eritque adeo $M - AS = 0$, ideoque $A = \frac{M}{S}$, posito $z = \frac{p}{q}$. Invento hoc pacto A, erit $M - AS$ divisibile per $p - qz$; ponatur ergo $\frac{M - AS}{p - qz} = T$, atque $T - BS - C(p-qz)S$ adhuc per $(p-qz)^2$ erit divisibile; fiet ergo $= 0$, posito $p - qz = 0$; ex quo prodit $B = \frac{T}{S}$ posito $z = \frac{p}{q}$. Sic autem invento B erit $T - BS$ divisibile per $p - qz$. Hanc ob rem, posito $\frac{T - BS}{p - qz} = V$, superest ut $V - CS$ divisibile sit per $p - qz$; eritque ergo $V - CS = 0$, posito $p - qz = 0$, atque $C = \frac{V}{S}$, posito $z = \frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C, fractiones partiales ex denominatoris N Factore $(p-qz)^3$ ortæ erunt $\frac{A}{(1-z)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$.

Cap. II.

Exemplum.

Sit proposita hæc fracta Functio $\frac{zz}{(1-z)^3(1+zz)}$, ex cujus denominatoris Factore cubico $(1-z)^3$ oriantur hæ fractiones partiales: $\frac{A}{(1-z)^3} + \frac{B}{(1-z)^2} + \frac{C}{1-z}$. Erit ergo $M = zz$ & $S = 1 + zz$; unde fit primum $A = \frac{zz}{1+zz}$

LIB. I. posito $1 - z = 0$ seu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam ponatur $T = \frac{M - AS}{1 - z}$; erit $T = \frac{\frac{1}{2}zz - \frac{1}{2}}{1 - z} = -\frac{1}{2} - \frac{1}{2}z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2}z}{1 + zz}$, posito $z = 1$, ita ut sit $B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2}S}{1 - z}$; erit $V = \frac{-\frac{1}{2}z + \frac{1}{2}zz}{1 - z} = -\frac{1}{2}z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1 + zz}$ posito $z = 1$, ita ut sit $C = -\frac{1}{4}$. Quo circa fractiones partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt

$$\frac{1}{2(1 - z)^3} - \frac{1}{2(1 - z)^2} - \frac{1}{4(1 - z)}.$$

45. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^n$; *fractiones partiales hinc ortæ* $\frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \ldots\ldots + \frac{K}{p - qz}$ *sequenti modo invenientur.*

Ponatur denominator $N = (p - qz)^n Z$, atque, ratiocinium ut ante instituendo, reperietur ut sequitur:

Primo $A = \frac{M}{Z}$, posito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$
Secundo $B = \frac{P}{Z}$, posito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$
Tertio $C = \frac{Q}{Z}$, posito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$
Quarto $D = \frac{R}{Z}$, posito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$
Quinto $E = \frac{S}{Z}$, posito $z = \frac{p}{q}$. &c.

Hoc ergo modo si definiantur singuli numeratores constan-
tes

tes A, B, C, D, &c. invenientur omnes fractiones partiales, quæ ex denominatoris N Factore $(p - qz)^n$ nascuntur.

Exemplum.

Sit proposita ista Functio fracta $\frac{1+zz}{z^5(1+z^3)}$ ex cujus denominatoris Factore z^5 nascantur hæ fractiones partiales $\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}$. Ad quarum numeratores constantes inveniendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; & $\frac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum est $A = \frac{M}{Z} = \frac{1+zz}{1+z^3}$, posito $z = 0$; ergo $A = 1$.

Ponatur $P = \frac{M - AZ}{z} = \frac{zz - z^3}{z} = z - zz$. Eritque secundo $B = \frac{P}{Z} = \frac{z - zz}{1+z^3}$, posito $z = 0$; ergo $B = 0$.

Ponatur $Q = \frac{P - BZ}{z} = \frac{z - zz}{z} = 1 - z$; eritque tertio $C = \frac{Q}{Z} = \frac{1-z}{1+z^3}$, posito $z = 0$: ergo $C = 1$.

Ponatur $R = \frac{Q - CZ}{z} = \frac{-z - z^3}{z} = -1 - zz$; erit quarto $D = \frac{R}{Z} = \frac{-1-zz}{1+z^3}$, posito $z = 0$; ex quo fit $D = -1$.

Ponatur $S = \frac{R - DZ}{z} = \frac{-zz + z^3}{z} = -z + zz$; erit quinto $E = \frac{S}{Z} = \frac{-z+zz}{1+z^3}$, posito $z = 0$; unde fit $E = 0$.

Quo circa fractiones partiales quæsitæ erunt hæ:

$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}$.

Lib. I. 46. *Quæcunque ergo proposita fuerit Functio rationalis fracta* $\frac{M}{N}$, *ea sequenti modo in partes resolvetur, atque in formam simplicissimam transmutabitur.*

Quærantur denominatoris N omnes Factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur & ex unoquoque per §. 41, fractio partialis eruatur. Quod si idem Factor simplex bis vel pluries occurrat, ii conjunctim sumantur atque ex eorum producto, quod erit potestas formæ $(p - qz)^n$, quærantur fractiones partiales convenientes per §. 45. Hocque modo cum ex singulis Factoribus simplicibus denominatoris erutæ fuerint fractiones partiales, tum harum omnium aggregatum æquabitur Functioni propositæ $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adjici debebit, quo prodeat valor Functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est sive fractiones partiales ante extractionem partis integræ, sive post quærantur. Eædem enim ex singulis denominatoris N Factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M, sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

Exemplum.

Quæratur valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma simplicissima expressus. Sumatur primum Factor denominatoris solitarius $1 + z$, qui dat $\frac{p}{q} = -1$, erit $M = 1$ & $Z = z^3 - 2z^4 + z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A = \frac{1}{z^3 - 2z^4 + z^5}$, posito $z = -1$; ideoque fit $A =$

$-\frac{1}{4}$, atque ex Factore $1+z$ oritur hæc fractio partialis $\frac{-1}{4(1+z)}$. Jam ſumatur Factor quadratus $(1-z)^2$ qui dat $\frac{p}{q}=1$, $M=1$, & $Z=z^3+z^4$. Poſitis ergo fractionibus partialibus hinc ortis $\frac{A}{(1-z)^2}+\frac{B}{1-z}$, erit $A=\frac{1}{z^3+z^4}$, poſito $z=1$; ergo $A=\frac{1}{2}$; fiat $P=\frac{M-\frac{1}{2}Z}{1-z}=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$; eritque $B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$, poſito $z=1$; ergo $B=\frac{7}{4}$ & fractiones partiales quæſitæ $\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$. Denique tertius Factor cubicus z^3 dat $\frac{p}{q}=0$; $M=1$; & $Z=1-z-zz+z^3$. Poſitis ergo fractionibus partialibus his $\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$; erit primum $A=\frac{M}{N}=\frac{1}{1-z-zz+z^3}$, poſito $z=0$; ergo $A=1$. Ponatur $P=\frac{M-Z}{z}=1+z-zz$, erit $B=\frac{P}{Z}$, poſito $z=0$; ergo $B=1$. Ponatur $Q=\frac{P-Z}{z}=2-zz$; erit $C=\frac{Q}{Z}$, poſito $z=0$; ergo $C=2$. Hanc ob rem Functio propoſita $\frac{1}{z^3(1-z)^2(1+z)}$ in hanc formam reſolvitur $\frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}$: nulla enim pars integra inſuper accedit, quia fractio propoſita non eſt ſpuria. Cap. II.

CAPUT III.

De transformatione Functionum per substitutionem.

46 bis. *Si fuerit* y *Functio quæcunque ipsius* z, *atque* z *definiatur per novam variabilem* x, *tum quoque* y *per* x *definiri poterit.*

Cum ergo antea y fuisset Functio ipsius z, nunc nova quantitas variabilis x inducitur, per quam utraque priorum y & z definiatur. Sic, si fuerit $y = \frac{1 - zz}{1 + zz}$, atque ponatur $z = \frac{1 - x}{1 + x}$; hoc valore loco z substituto, erit $y = \frac{2x}{1 + xx}$. Sumpto ergo pro x valore quocunque determinato, ex eo reperientur valores determinati pro z & y, sicque invenitur valor ipsius y respondens illi valori ipsius z qui simul prodiit. Uti si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1 - zz}{1 + zz}$, cui expressioni y æquatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem hæc novæ variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expessio ipsius y per z data laborat, tollitur; vel quando ob æquationem altioris gradus, qua relatio inter y & z exprimitur, non licet Functionem explicitam ipsius z ipsi y æqualem exhibere, nova variabilis x introducitur, ex qua utraque y & z commode definiri queat: unde insignis substitutionum usus jam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit* $y = \sqrt{(a + bz)}$; *nova variabilis* x *per quam utraque* z & y *rationaliter exprimatur, sequenti modo invenietur.*

Quoniam tam z quam y debet esse Functio rationalis ipsius x; perspicuum est hoc obtineri si ponatur $\sqrt{(a + bz)} = bx$: Fiet enim primo $y = bx$; & $a + bz = bbxx$; hincque $z = bxx - \frac{a}{b}$. Quocirca utraque quantitas y & z per Functionem rationalem ipsius x exprimitur; scilicet cum sit $y = \sqrt{(a + bz)}$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

48. *Si fuerit* $y = (a + bz)^{m:n}$; *nova variabilis* x, *per*
quam tam y *quam* z *rationaliter exprimatur, ſic reperietur.*

Ponatur $y = x^m$, fietque $(a + bz)^{m:n} = x^m$ ideoque $(a + bz)^{1:n} = x$: ergo $a + bz = x^n$ & $z = \frac{x^n - a}{b}$. Sic ergo utraque quantitas y & z rationaliter per x definietur, ope ſcilicet ſubſtitutionis $z = \frac{x^n - a}{b}$, quæ præbet $y = x^m$. Quamvis igitur neque y per z, neque viciſſim z per y rationaliter exprimi poſſit; tamen utraque reddita eſt Functio rationalis novæ quantitatis variabilis x per ſubſtitutionem introductæ, ſcopo ſubſtitutionis omnino convenienter.

49. *Si fuerit* $y = (\frac{a + bz}{f + gz})^{m:n}$; *requiritur nova quantitas variabilis* x *per quam utraque* y & z *rationaliter exprimatur.*

Manifeſtum primo eſt ſi ponatur $y = x^m$, quæſito ſatisfieri; erit enim $(\frac{a + bz}{f + gz})^{m:n} = x^m$, ideoque $\frac{a + bz}{f + gz} = x^n$; ex qua æquatione elicitur $z = \frac{a - fx^n}{gx^n - b}$; quæ ſubſtitutio præbet $y = x^m$. Hinc quoque intelligitur ſi fuerit $(\frac{\alpha + \beta y}{\gamma + \delta y})^n = (\frac{a + bz}{f + gz})^m$, tam y quam z rationaliter per x expreſſum iri, ſi utraque formula ponatur $= x^{mn}$; reperietur enim $y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$ & $z = \frac{a - fx^n}{gx^n - b}$: qui caſus nil habent difficultatis.

50. *Si fuerit* $y = \surd((a + bz)(c + dz))$; *ſubſtitutio idonea invenietur, qua* y & z *rationaliter exprimantur, hoc modo.*

Ponatur $\surd((a + bz)(c + dz)) = (a + bz)x$, facile enim perſpicitur hinc valorem rationalem pro z eſſe proditurum; quia valor ipſius z per æquationem ſimplicem determinatur. Erit

 ergo $c + dz = (a + bz)\,xx$, hincque $z = \frac{c - axx}{bxx - d}$. Quare porro fiet $a + bz = \frac{bc - ad}{bxx - d}$; & ob $y = \sqrt{((a + bz)(c + dz))} = (a + bz)\,x$ habebitur $y = \frac{(bc - ad)\,x}{bxx - d}$. Functio ergo irrationalis $y = \sqrt{((a + bz)\,(c + dz))}$ ad rationalitatem perducitur ope substitutionis $z = \frac{c - axx}{bxx - d}$, quippe quæ dabit $y = \frac{(bc - ad)\,x}{bxx - d}$. Sic, si fuerit $y = \sqrt{(aa - zz)} = \sqrt{((a + z)\,(a - z))}$; ob $b = +1$, $c = a$, $d = -1$; ponatur $z = \frac{a - axx}{1 + xx}$, eritque $y = \frac{2ax}{1 + xx}$. Quoties ergo quantitas post signum $\sqrt{}$ habuerit duos Factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem Factores bini simplices fuerint imaginarii, sequenti modo uti præstabit.

51. *Sit* $y = \sqrt{(p + qz + rzz)}$; *atque requiritur substitutio idonea pro* z *facienda, ut valor ipsius* y *fiat rationalis.*

Pluribus modis hoc fieri potest, prout p & q fuerint quantitates vel affirmativæ vel negativæ. Sit primo p quantitas affirmativa, ac ponatur aa pro p; etiamsi enim p non sit quadratum, tamen irrationalitas quantitatum constantium præsens negotium non turbat. Sit igitur

I. $y = \sqrt{(aa + bz + czz)}$; ac ponatur $\sqrt{(aa + bz + czz)} = a + xz$; erit $b + cz = 2ax + xxz$; unde fit $z = \frac{b - 2ax}{xx - c}$: tum vero erit $y = a + xz = \frac{bx - axx - ac}{xx - c}$; ubi z & y sunt Functiones rationales ipsius x. Sit jam

II. $y = \sqrt{(aazz + bz + c)}$; ac ponatur $\sqrt{(aazz + bz + c)} = az + x$; erit $bz + c = 2axz + xx$, & $z = \frac{xx - c}{b - 2ax}$. Tum autem fit $y = az + x = \frac{-ac + bx - axx}{b - 2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, nisi sit $qq > 4pr$, valor ipsius y semper erit imaginarius. Quod si autem fuerit $qq > 4pr$; expressio $p + qz + rzz$ in duos Factores

resolvi poterit, qui casus ad §. praeced. reducitur. Saepenumero autem commodius ad hanc formam reducitur, $y = \sqrt{(aa + (b + cz)(d + ez))}$; pro qua ad rationalitatem perducenda ponatur $y = a + (b + cz)x$, eritque $d + ez = 2ax + bxx + cxxz$; unde fit $z = \frac{d - 2ax - bxx}{cxx - e}$, & $y = \frac{-ae + (cd - be)x - acxx}{cxx - e}$. Interdum commodius fieri potest reductio ad hanc formam, $y = \sqrt{(aazz + (b + cz)(d + ez))}$. Tum ponatur $y = az + (b + cz)x$; erit $d + ez = 2axz + bxx + cxxz$ & $z = \frac{bxx - d}{e - 2ax - cxx}$, atque $y = \frac{-ad + (be - cd)x - abxx}{e - 2ax - cxx}$. CAP. III.

EXEMPLUM.

Si habeatur ista ipsius z Functio irrationalis $y = \sqrt{(-1 + 3z - zz)}$; quae cum reduci queat ad hanc formam $y = \sqrt{(1 - 2 + 3z - zz)} = \sqrt{(1 - (1 - z)(2 - z))}$; ponatur $y = 1 - (1 - z)x$, erit $-2 + z = -2x + xx - xxz$ & $z = \frac{2 - 2x + xx}{1 + xx}$. Deinde est $1 - z = \frac{-1 + 2x}{1 + xx}$ & $y = 1 - (1 - z)x = \frac{1 + x - xx}{1 + xx}$. Atque hi sunt fere casus, quos Algebra indeterminata, seu methodus *Diophantaea*, suppeditat; neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. *Si y ejusmodi fuerit Functio ipsius z ut sit* $ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0$, *invenire novam variabilem x, per quam valores ipsarum y & z explicite assignari queant.*

Quoniam resolutio aequationum generalis non habetur, ex aequatione proposita $ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0$ neque y per z

LIB. I. neque viciſſim z per y exhiberi poteſt. Quo igitur huic incommodo remedium afferatur; ponatur $y = x^m z^n$, eritque $ax^{\alpha m} z^{\alpha n} + b z^{\epsilon} + c x^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc exponens n ita ut ex hac æquatione valor ipſius z definiri queat: quod tribus modis præſtari poteſt.

I. Sit $\alpha n = \epsilon$; ideoque $n = \frac{\epsilon}{\alpha}$; erit, æquatione per $z^{\alpha n} = z^{\epsilon}$ diviſa, $ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \epsilon + \delta} = 0$; unde oritur $z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{1}{\gamma n - \epsilon + \delta}}$, ſive

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\alpha}{\epsilon\gamma - \alpha\epsilon + \alpha\delta}}, \&$$

$$y = x^m \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\epsilon}{\epsilon\gamma - \alpha\epsilon + \alpha\delta}}.$$

II. Sit $\epsilon = \gamma n + \delta$ ſeu $n = \frac{\epsilon - \delta}{\gamma}$; erit, æquatione per z^{ϵ} diviſa, $ax^{\alpha m} z^{\alpha n - \epsilon} + b + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{1}{\alpha n - \epsilon}} = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\gamma}{\alpha\epsilon - \alpha\delta - \epsilon\gamma}},$$

atque $y = x^m \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\epsilon - \delta}{\alpha\epsilon - \alpha\delta - \epsilon\gamma}}$.

III. Sit $\alpha n = \gamma n + \delta$, ſeu $n = \frac{\delta}{\alpha - \gamma}$; erit, æquatione per $z^{\alpha n}$ diviſa, $ax^{\alpha m} + b z^{\epsilon - \alpha n} + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{1}{\epsilon - \alpha n}} =$$

$(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b})^{\frac{\alpha-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}}$; atque

$$y=x^m\left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta-\beta\gamma-\alpha\delta}}.$$

Tribus igitur diversis modis erutæ sunt Functiones ipsius x, quæ ipsis z & y sunt æquales. Præterea vero pro m numerum pro lubitu substituere licet cyphra excepta; sicque formulæ ad commodissimam expressionem reduci poterunt.

Exemplum.

Exprimatur natura Functionis y per hanc æquationem $y^3+z^3-cyz=0$; atque quærantur Functiones ipsius x ipsis y & z æquales. Erit ergo $a=-1$; $b=-1$; $\alpha=3$; $\beta=3$; $\gamma=1$; & $\delta=1$. Hinc primus modus dabit, posito $m=1$, $z=(\frac{x^3+1}{cx})^{-1}$ & $y=x(\frac{x^3+1}{cx})^{-1}$, sive $z=\frac{cx}{1+x^3}$ & $y=\frac{cxx}{1+x^3}$; quarum expressionum utraque adeo est rationalis.

Secundus modus vero dabit hos valores:

$$z=\left(\frac{cx-1}{x^3}\right)^{1:3},\ \&\ y=x\left(\frac{cx-1}{x^3}\right)^{2:3},\ \text{sive}$$

$$z=\frac{1}{x}\sqrt[3]{(cx-1)},\ \&\ y=\frac{1}{x}\sqrt[3]{(cx-1)^2}.$$

Tertius modus ita rem expediet ut sit

$$z=(cx-x^3)^{1:3},\ \&\ y=x\ (cx-x^3)^{1:3}.$$

53. *Hinc à posteriori intelligitur cujusmodi æquationes, quibus valor Functionis* y *per* z *determinatur, hoc modo novam variabilem* x *introducendo resolvi queant.*

Ponamus enim resolutione jam instituta prodiisse has deter-

LIB. I. minationes $z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{p:r}$, atque $y = x \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{q:r}$; eritque $y^p = x^p z^q$; & hinc $x = yz^{-q:p}$. Cum igitur sit $z^{r:p} = \frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}$, si loco x ejus valorem $yz^{-q:p}$ substituamus; prodibit ista æquatio $z^{r:p} = \frac{ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.}{A + By^\mu z^{-\mu q:p} + Cy^\nu z^{-\nu q:p} + \&c.}$; quæ reducitur ad hanc $Az^{r:p} + By^\mu z^{(r-\mu q):p} + Cy^\nu z^{(r-\nu q):q} + \&c. = ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.$ quæ multiplicata per $z^{\alpha q:p}$ transibit in hanc: $Az^{(\alpha q+r):p} + By^\mu z^{(\alpha q-\mu q+r):p} + Cy^\nu z^{(\alpha q-\nu q+r):p} + \&c. = ay^\alpha + by^\beta z^{(\alpha q-\beta q):p} + cy^\gamma z^{(\alpha q-\gamma q):p} + \&c.$ Ponatur $\frac{\alpha q + r}{p} = m$ & $\frac{\alpha q - \beta q}{p} = n$: fiet [*] $p = \alpha - \beta$; $q = n$, & $r = \alpha m - \beta m - \alpha n$: atque nascetur ista æquatio: $Az^m + By^\mu z^{m-\mu n:(\alpha-\beta)} + Cy^\nu z^{m-\nu n:(\alpha-\beta)} + \&c. = ay^\alpha + by^\beta z^n + cy^\gamma z^{(\alpha-\gamma)n:(\alpha-\beta)} + \&c.$ quæ propterea ita resolvetur ut sit:

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\alpha-\beta}{\alpha m - \beta m - \alpha n}} \ \&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{n}{\alpha m - \beta m - \alpha n}}.$$

Vel ponatur $\frac{\alpha q + r}{p} = m$, & $\frac{\alpha q - \mu q + r}{p} = n$, erit $m - n$

[*] Quoniam tres adsunt indeterminatæ, & duæ tantum æquationes, licuit ponere $p = \alpha - \beta$.

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m - n}{\mu}$, atque $\frac{r}{p} = m - \frac{\alpha m + \alpha n}{\mu}$. Hinc CAP. III.
fit [*] $p = \mu$; $q = m - n$; & $r = \mu m - \alpha m + \alpha n$; atque hæc æquatio reſultabit:

$$A z^m + B y^\mu z^n + C y^\nu z^{m - \nu(m-n):\mu} + \&c. = a y^\alpha + b y^\beta z^{(\alpha - \beta)(m-n):\mu} + c y^\gamma z^{(\alpha-\gamma)(m-n):\mu} + \&c.$$

quæ ita reſolvetur ut ſit:

$$z = \left(\frac{a x^\alpha + b x^\beta + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.}\right)^{\frac{\mu}{\mu m - \alpha m + \alpha n}} \&$$

$$y = x \left(\frac{a x^\alpha + b x^\beta + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.}\right)^{\frac{m-n}{\mu m - \alpha m + \alpha n}}.$$

54. *Si* y *ita pendeat a* z *ut ſit* $ayy + byz + czz + dy + ez = 0$, *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, erit diviſione per z facta: $axxz + bxz + cz + dx + e = 0$, ex qua reperitur $z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propoſitam reduci poteſt hæc æquatio inter y & z: $ayy + byz + czz + dy + ez + f = 0$ diminuendo vel augendo utramque variabilem certa quadam quantitate conſtante, unde & hæc æquatio per novam variabilem x rationaliter explicari poteſt.

55. *Si* y *ita pendeat a* z, *ut ſit* $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0$; *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimi poterit.*

Ponatur $y = xz$, & facta ſubſtitutione tota æquatio per zz dividi poterit: prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = 0$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$ ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

[*] Eadem cauſa qua ſupra fuit $p = \alpha - \beta$.

Ex his casibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ esse debeant, ut hujusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujusmodi casus sæpius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. *Si* y *ita pendeat a* z *ut sit* $ayy + byz + czz = d$ *hoc modo utraque quantitas* y & z *per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\frac{d}{axx + bx + c}}$ & $y = x\sqrt{\frac{d}{axx + bx + c}}$.

Simili modo si fuerit, $ay^3 + by^2 z + cyz^2 + dz^3 = ey + fz$; posito $y = xz$, tota æquatio per z divisa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$; & $y = x\sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$. Hi autem casus aliique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. *Si* y *ita pendeat a* z *ut sit* $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$, *sequenti modo tam* z *quam* y *commode per novam variabilem* x *exprimetur.*

Sit $y = xz$, atque facta substitutione tota æquatio dividi poterit per z^n, siquidem exponens m sit major quam n; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$$z = \left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^{m} + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)} \& \quad \text{Cap.III.}$$

$$y = x\left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^{m} + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)}$$

Hæc ſcilicet reſolutio locum habet, ſi in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimenſionum ab y & z ſumptarum numerus; uti in caſu tractato in ſingulis terminis numerus dimenſionum vel eſt m vel n.

58. *Si in æquatione inter* y & z *triplicis generis dimenſiones occurrant, quarum ſumma tantum ſuperet mediam, quantum hæc media infimam, ope reſolutionis æquationis quadratæ variabiles* y & z *per novam* x *exprimi poterunt.*

Si enim ponatur $y = xz$, diviſione per minimam ipſius z poteſtatem facta, valor ipſius z per x, ope extractionis radicis quadratæ exhibebitur, id quod ex ſequentibus exemplis erit manifeſtum.

Exemplum I.

Sit $ay^3 + by^2z + cyzz + dz^3 = 2eyy + 2fyz + 2gzz + hy + iz$; ac ponatur $y = xz$: erit, diviſione per z facta, $(ax^3 + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i$; ex qua ſequens ipſius z obtinebitur valor:

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^2 + (ax^3 + bxx + cx + d)(hx + i))}}{ax^3 + bxx + cx + d},$$

quo invento erit $y = xz$.

Exemplum II.

Sit $y^5 = 2az^3 + by + cz$; ac, poſito $y = xz$, erit $x^5 z^4 = 2azz + bx + c$; ex qua reperitur $zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$; &

$$z = \frac{\pm\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{xx\sqrt{x}} \; \& \; y = \frac{\pm\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{x\sqrt{x}}.$$

EXEMPLUM III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimenſiones ſint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 diviſa abibit in hanc: $x^{10} z^6 = 2axz^3 + bx + c$ ſeu $z^6 = \frac{2axz^3 + bx + c}{x^{10}}$; unde invenitur $z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$; ideoque erit $z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$; atque $y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^2}$. Ex quibus exemplis uſus hujuſmodi ſubſtitutionum abunde perſpicitur.

CAPUT IV.

De explicatione Functionum per ſeries infinitas.

59. CUM Functiones fractæ atque irrationales ipſius z non in forma integra $A + Bz + Cz^2 + Dz^3 + \&c.$ continentur, ita ut terminorum numerus ſit finitus, quæri ſolent hujuſmodi expreſſiones in infinitum excurrentes, quæ valorem cujuſvis Functionis ſive fractæ ſive irrationalis exhibeant. Quin etiam natura Functionum tranſcendentium melius intelligi cenſetur, ſi per ejuſmodi formam, etſi infinitam, exprimantur. Cum enim natura Functionis integræ optime perſpiciatur, ſi ſecundum diverſas poteſtates ipſius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 + \&c.$ reducatur, ita eadem forma aptiſſima videtur ad reliquarum Functionum omnium indolem menti repræſentandam, etiamſi terminorum numerus ſit revera infinitus. Perſpicuum autem eſt nullam Functionem non integram ipſius z per numerum hujuſmodi terminorum $A + Bz + Cz^2 + \&c.$ finitum exponi poſſe; eo ipſo enim

Functio foret integra; num vero per hujusmodi terminorum seriem infinitam exhiberi possit, si quis dubitet, hoc dubium per ipsam evolutionem cujusque Functionis tolletur. Quo autem hæc explicatio latius pateat, præter potestates ipsius z exponentes integros affirmativos habentes, admitti debent potestates quæcunque. Sic dubium erit nullum quin omnis Functio ipsius z in hujusmodi expressionem infinitam transmutari possit: $Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} +$ &c. denotantibus exponentibus α, β, γ, δ, &c. numeros quoscunque. CAP. IV.

60. *Per divisionem autem continuam intelligitur fractionem* $\frac{a}{\alpha + \beta z}$ *resolvi in hanc seriem infinitam* $\frac{a}{\alpha} - \frac{a\beta z}{\alpha^2} + \frac{a\beta^2 z^2}{\alpha^3} - \frac{a\beta^3 z^3}{\alpha^4} + \frac{a\beta^4 z^4}{\alpha^5} -$ *&c., quæ, cum quilibet terminus ad sequentem habeat rationem constantem* $1 : \frac{\beta z}{\alpha}$, *vocatur series geometrica.*

Potest vero quoque hæc series ita inveniri, ut ipsa initio pro incognita habeatur: ponatur enim $\frac{a}{\alpha + \beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 +$ &c. atque ad æqualitatem producendam quærantur coëfficientes A, B, C, D, &c. Erit ergo $a = (\alpha + \beta z)(A + Bz + Cz^2 + Dz^3 +$ &c.$)$, & multiplicatione actu peracta fiet

$$\begin{aligned} a = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \end{aligned}$$

Quamobrem esse debet $a = \alpha A$, ideoque $A = \frac{a}{\alpha}$, & coëfficientium uniuscujusque potestatis ipsius z summa nihilo æqualis est ponenda: unde prodibunt hæ æquationes,

$\alpha B + \beta A = 0$ cognito ergo quovis coëfficiente
$\alpha C + \beta B = 0$ facile reperitur sequens; si enim
$\alpha D + \beta C = 0$ fuerit coëfficiens termini cujusque $= P$
$\alpha E + \beta D = 0$ & sequens $= Q$ erit $\alpha Q + \beta P = 0$
&c. sive $Q = \frac{\beta P}{\alpha}$.

LIB. I. Cum igitur terminus primus A fit determinatus $= \frac{a}{\alpha}$ ex eo fequentes litteræ B, C, D, &c. definiuntur eodem modo, quo ex divifione funt orti. Ceterum ex infpectione perfpicitur in ferie infinita pro $\frac{a}{\alpha + \beta z}$ inventa poteftatis z^n coëfficientem fore $= \pm \frac{a \beta^n}{\alpha^{n+1}}$, ubi fignum $+$ locum habet fi n fit numerus par, fignum $-$ autem fi n fit numerus impar: feu coëfficiens erit $= \frac{a}{\alpha} \left(\frac{-\beta}{\alpha} \right)^n$.

61. *Simili modo ope divifionis continuatæ hæc Functio fracta* $\frac{a + bz}{\alpha + \beta z + \gamma zz}$ *in feriem infinitam converti poteft.*

Cum autem divifio fit tædiofa, neque tam facile naturam feriei infinitæ oftendat, commodius erit feriem quæfitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \beta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

multiplicetur utrinque per $\alpha + \beta z + \gamma zz$, atque fiet

$$\begin{aligned} a + bz = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \&c. \end{aligned}$$

Hinc erit $\alpha A = a$; $\alpha B + \beta A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a\beta}{\alpha\alpha}$; reliquæ vero litteræ ex fequentibus æquationibus determinabuntur:

$\alpha C + \beta B + \gamma A = 0$
$\alpha D + \beta C + \gamma B = 0$
$\alpha E + \beta D + \gamma C = 0$
$\alpha F + \beta E + \gamma D = 0$
&c.

hinc ergo ex binis quibufque coëfficientibus contiguis fequens reperitur. Sic fi duo coëfficientes contigui fuerint P, Q & fequens R, erit $\alpha R + \beta Q + \gamma P = 0$ feu $R = \frac{-\beta Q - \gamma P}{\alpha}$.

Cum igitur duæ litteræ primæ A & B jam fint inventæ fequentes C, D, E, F &c. omnes fucceffive ex iis invenientur,

tur, sicque reperietur Series infinita $A + Bz + Cz^2 + Dz^3 + \&c.$
Functioni fractæ propositæ $\frac{a + bz}{\alpha + \beta z + \gamma z^2}$ æqualis.

EXEMPLUM.

Si fuerit proposita hæc fractio $\frac{1 + 2z}{1 - z - zz}$, huicque æqualis statuatur Series $A + Bz + Cz^2 + Dz^3 + \&c.$ ob $a = 1$; $b = 2$; $\alpha = 1$; $\beta = -1$; $\gamma = -1$; erit $A = 1$; $B = 3$; tum vero erit

$C = B + A$ quilibet ergo coëfficiens æqualis est sum-
$D = C + B$ mæ duorum præcedentium; quare si cog-
$E = D + C$ niti fuerint duo coëfficientes contigui
$F = E + D$ P & Q, erit sequens $R = P + Q$.
&c.

Cum igitur duo coëfficientes primi A & B sint cogniti, fractio proposita $\frac{1 + 2z}{1 - z - zz}$ in hanc Seriem infinitam transmutatur $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \&c.$, quæ nullo negotio quousque libuerit continuari potest.

62. Ex his jam satis intelligitur indoles Serierum infinitarum, in quas Functiones fractæ transmutantur; tenent enim ejusmodi legem, ut quilibet terminus ex aliquot præcedentibus determinari possit. Scilicet, si denominator fractionis propositæ fuerit $\alpha + \beta z$, atque Series infinita statuatur

$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \&c.$;

quilibet coëfficiens Q ex præcedente P solo ita definietur ut sit $\alpha Q + \beta P = 0$. Sin denominator fuerit trinomium $\alpha + \beta z + \gamma zz$, quilibet Seriei coëfficiens R ex duobus præcedentibus Q & P ita definietur ut sit $\alpha R + \beta Q + \gamma P = 0$: simili modo si denominator fuerit quadrinomium, ut $\alpha + \beta z + \gamma zz + \delta z^3$, quilibet seriei coëfficiens S ex tribus antecedentibus R, Q & P ita determinabitur, ut sit $\alpha S + \beta R + \gamma Q + \delta P = 0$,

sicque de ceteris. In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus secundum legem quandam constantem, quæ lex ex denominatore fractionis hanc Seriem producentis sponte apparet. Vocari autem hæ Series a Celeb. Moivræo, qui earum naturam maxime est scrutatus, solent *recurrentes*, propterea quod ad terminos antecedentes est recurrendum, si sequentes investigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus constans α non sit $= 0$: cum enim inventus sit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is, tum omnes sequentes fierent infiniti, si esset $\alpha = 0$. Hoc ergo casu excluso, quem deinceps evolvam, Functio fracta in Seriem infinitam recurrentem transmutanda, hujusmodi habebit formam $\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim semper fractio reduci potest, nisi is sit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut Seriei hinc formatæ omnes termini fiant affirmativi. Quod si enim Series recurrens hinc orta ponatur $A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ coëfficientes ita determinabuntur ut sit

$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + \beta A + c$$
$$D = \alpha C + \beta B + \gamma A + d$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
$$\&c.$$

Quilibet ergo coëfficiens æqualis est aggregato ex multiplis aliquot præcedentium una cum numero quodam, quem numerator præbet. Nisi autem numerator in infinitum progrediatur, hæc additio mox cessabit, atque quivis terminus secundum legem constantem ex aliquot præcedentibus determinabitur. Ne ergo lex progressionis usquam turbetur conveniet

Functionem fractam genuinam adhibere : si enim fractio spuria accipiatur, tum pars integra in ea contenta ad Seriem accedet, atque in illis terminis, quos vel auget vel minuit, legem progressionis interrumpet. Exempli gratia hæc fractio spuria $\frac{1 + 2z - z^3}{1 - z - zz}$, præbebit hanc Seriem $1 + 3z + 4zz + 6z^3 + 10z^4 + 16z^5 + 26z^6 + 42z^7 +$ &c. ubi a lege, qua quivis coëfficiens est summa duorum præcedentium, terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes merentur, si denominator fractionis, unde oriuntur, fuerit potestas. Sic, si ista fractio $\frac{a + bz}{(1 - \alpha z)^2}$ in Seriem resolvatur, prodit

$$\begin{array}{llllll} a & + 2\alpha a & + 3\alpha^2 a & + 4\alpha^3 a & + 5\alpha^4 a & + \&c. \\ & z & z^2 & z^3 & z^4 & \\ + & b & + 2\alpha b & + 3\alpha^2 b & + 4\alpha^3 b & \end{array}$$

in qua coëfficiens potestatis z^n erit $(n + 1)\alpha^n a + n\alpha^{n-1} b$. Erit tamen hæc Series recurrens, quia quilibet terminus ex duobus præcedentibus determinatur, cujus determinationis lex perspicitur ex denominatore evoluto $1 - 2\alpha z + \alpha\alpha zz$. Si ponatur $\alpha = 1$ & $z = 1$, abit Series in progressionem arithmeticam generalem $a + (2a + b) + (3a + 2b) + (4a + 3b) +$ &c. cujus differentiæ sunt constantes. Omnis ergo progressio arithmetica est Series recurrens : si enim sit

$A + B + C + D + E + F +$ &c. progressio arithmetica, erit $C = 2B - A$; $D = 2C - B$; $E = 2D - C$, &c.

65. Deinde hæc fractio $\frac{a + bz + czz}{(1 - \alpha z)^3}$ ob $\frac{1}{(1 - \alpha z)^3} = (1 - \alpha z)^{-3} = 1 + 3\alpha z + 6\alpha^2 z^2 + 10\alpha^3 z^3 + 15\alpha^4 z^4 +$ &c. transmutabitur in hanc Seriem infinitam :

$$\begin{array}{llllll} a & + 3\alpha a & + 6\alpha^2 a & + 10\alpha^3 a & + 15\alpha^4 a & + \&c. \\ & z & & & & \\ + & b & + 3\alpha b z^2 & + 6\alpha^2 b z^3 & + 10\alpha^3 b z^4 & \\ & & + c & + 3\alpha c & + 6\alpha^2 c & \end{array}$$

LIB. I. in qua potestas z^n coëfficientem habebit $\frac{(n+1)(n+2)}{1.\ 2}\alpha^n a + \frac{n(n+1)}{1.\ 2}\alpha^{n-1} b + \frac{(n-1)n}{1.\ 2}\alpha^{n-2} c$. Quod si autem ponatur $\alpha = 1$ & $z = 1$, Series hæc abibit in progressionem generalem secundi ordinis, cujus differentiæ secundæ sunt constantes. Designet $A + B + C + D + E +$ &c. hujusmodi progressionem, erit ea simul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut sit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progressione arithmetica procedentium secundæ differentiæ quoque sint æquales, nempe $= 0$, hæc proprietas quoque ad progressiones arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + czz + dz^3}{(1 - \alpha z)^4}$ dabit Seriem infinitam, in qua potestas ipsius z quæcunque z^n hunc habebit coëfficientem $\frac{(n+1)(n+2)(n+3)}{1.\ 2.\ 3}\alpha^n a + \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\alpha^{n-1} b + \frac{(n-1)n(n+1)}{1.\ 2.\ 3}\alpha^{n-2} c + \frac{(n-2)(n-1)n}{1.\ 2.\ 3}\alpha^{n-3} d$: posito ergo $\alpha = 1$ & $z = 1$, hæc Series in se complectetur omnes progressiones algebraicas tertii ordinis, quarum differentiæ tertiæ sunt constantes: omnes ergo hujus ordinis progressiones, cujusmodi sit $A + B + C + D + E + F +$ &c. erunt simul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortæ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas simul in omnes progressiones inferiorum ordinum competit.

67. Hoc modo ostendetur omnes progressiones algebraicas cujuscunque ordinis, quæ tandem ad differentias constantes deducunt, esse Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, existente n numero majore quam is, qui ordinem progressionis indicat. Cum igitur $a^m + (a+b)^m +$

$(a+2b)^m + (a+3b)^m +$ &c. exhibeat progreſſionem ordinis m; erit ob naturam Serierum recurrentium

$$0 = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1.\ 2}(a+2b)^m - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}(a+3b)^m + \ldots\ldots\ldots\ldots \pm \frac{n}{1}(a+(n-1)b)^m \mp (a+nb)^m;$$

ubi ſigna ſuperiora valent ſi n ſit numerus par, inferiora autem ſi n ſit numerus impar. Hæc ergo æquatio ſemper eſt vera ſi fuerit n numerus integer major quam m. Hinc ergo intelligitur quam late pateat doctrina de Seriebus recurrentibus.

68. Si denominator fuerit poteſtas non binomii ſed multinomii, natura Seriei quoque alio modo explicari poteſt. Sit nempe hæc fractio $\frac{1}{(1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{\&c.})^{m+1}}$ propoſita, erit Series infinita hinc nata

$$1 + \frac{(m+1)}{1}\alpha z + \left.\begin{array}{l} \frac{(m+1)(m+2)}{1.\ 2}\alpha^2 \\ + \frac{(m+1)}{1}\beta \end{array}\right\} z^2 + \left.\begin{array}{l} \frac{(m+1)(m+2)(m+3)}{1.\ 2.\ 3}\alpha^3 \\ + \frac{(m+1)(m+2)}{1.\ 2}2\alpha\beta \\ + \frac{(m+1)}{1}\gamma \end{array}\right\} z^3 + \text{\&c.}$$

Ad naturam hujus Seriei penitus inſpiciendam, exponatur hæc Series per litteras generales hoc modo:

$1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n +$ &c., ac quilibet coëfficiens N ex tot procedentibus, quot ſunt litteræ $\alpha, \beta, \gamma, \delta$, &c. ita determinabitur ut ſit: $N = \frac{m+n}{n}\alpha M + \frac{2m+n}{n}\beta L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I +$ &c. quæ lex continuationis etſi non eſt conſtans, ſed ab exponente poteſtatis z pendet, tamen eidem Seriei alia convenit lex progreſſionis conſtans, quam denominator evolutus præbet,

LIB. I. naturæ Serierum recurrentium consentaneam. Illa vero lex non constans tantum locum habet si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus posuimus primum denominatoris terminum constantem non esse $= 0$, ejusque loco unitatem collocavimus; nunc videamus cujusmodi Series oriantur, si in denominatore terminus constans evanescat. His casibus ergo Functio fracta hujusmodi formam habebit $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta zz - \&c.)}$, convertatur ergo, neglecto denominatoris Factore z, reliqua fractio $\frac{a + bz + czz + \&c.}{1 - \alpha z - \beta z^2 - \&c.}$ in Seriem recurrentem $A + Bz + Cz^2 + Dz^3 + \&c.$ atque, facta divisione per z, manifestum est fore $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \&c.$ Simili modo erit $\frac{a + bz + cz^2 + \&c.}{z^2(1 - \alpha z - \beta z^2 - \&c.)} = \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \&c.$, atque generatim erit

$$\frac{a + bz + cz^2 + \&c.}{z^m(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \&c.$$

quicunque numerus fuerit exponens m.

70. Quoniam per substitutionem loco z alia variabilis x in Functionem fractam introduci, hocque pacto Functio fracta quævis in innumerabiles formas diversas transmutari potest; hoc modo eadem Functio fracta infinitis modis per Series recurrentes explicari poterit. Sit scilicet proposita hæc fractio $y = \frac{1 + z}{1 - z - zz}$ & per Seriem recurrentem $y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \&c.$: ponatur $z = \frac{1}{x}$ erit y

$= \frac{xx+x}{xx-x-1} = \frac{-x(1+x)}{1+x-xx}$. Jam $\frac{1+x}{1+x-xx} = 1 +$
$0x + xx - x^3 + 2x^4 - 3x^5 + 5x^6 -$ &c.; unde erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$ &c. Vel ponatur $z = \frac{1-x}{1+x}$, erit $y = \frac{-2-2x}{1-4x-xx}$; unde fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 -$ &c. cujusmodi Series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali in Series infinitas transformari solent, quod sit $(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n.\ 2n} P^{\frac{m-2n}{n}} Q^2 + \frac{m(m-n)(m-2n)}{n.\ 2n.\ 3n} P^{\frac{m-3n}{n}} Q^3 +$ &c.: hi enim termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m & n numeros definitos scribendo.

$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1.\ 1}{2.\ 4} P^{-\frac{3}{2}} Q^2 + \frac{1.\ 1.\ 3}{2.\ 4.\ 6} P^{-\frac{5}{2}} Q^3 -$ &c.

$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1.\ 3}{2.\ 4} P^{-\frac{5}{2}} Q^2 - \frac{1.\ 3.\ 5}{2.\ 4.\ 6} P^{-\frac{7}{2}} Q^3 +$ &c.

$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1.\ 2}{3.\ 6} P^{-\frac{5}{3}} Q^2 + \frac{1.\ 2.\ 5}{3.\ 6.\ 9} P^{-\frac{8}{3}} Q^3 -$ &c.

$$(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1.\,4}{3.\,6} P^{-\frac{7}{3}} Q^2 - \frac{1.\,4.\,7}{3.\,6.\,9} P^{-\frac{10}{3}} Q^3 + \&c.$$

$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2.\,1}{3.\,6} P^{-\frac{4}{3}} Q^2 + \frac{2.\,1.\,4}{3.\,6.\,9} P^{-\frac{7}{3}} Q^3 - \&c.$$

&c.

72. Hujusmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari possit: sit enim Seriei, quæ ex $(P+Q)^{\frac{m}{n}}$ nascitur, terminus quilibet $= MP^{\frac{m-kn}{n}} Q^k$ erit sequens $= \frac{m-kn}{(k+1)n} MP^{\frac{m-(k+1)n}{n}} Q^{k+1}$. Notandum autem est in quovis termino sequente exponentem ipsius P unitate decrescere, contra vero exponentem ipsius Q unitate crescere. Quo autem hæc facilius ad quemvis casum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi potest $P^{\frac{m}{n}}(1+\frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1+\frac{Q}{P})^{\frac{m}{n}}$ Serieque resultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipsa Series ante data. Tum vero si m non solum numeros integros denotet, sed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, si pro $\frac{Q}{P}$, quæ est Functio ipsius z, ponatur Z, habebitur $(1+Z)^m = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1.\,2} Z^2 + \frac{m(m-1)(m-2)}{1.\,2.\,3} Z^3 + \&c.$ Ad sequentes progressionum leges autem observandas conveniet hanc formulæ generalis in Seriem conversionem notasse

taſſe $(1+Z)^{m-1} = 1 + \frac{(m-1)}{1} Z + \frac{(m-1)}{1.} \frac{(m-2)}{2} Z^2 + \frac{(m-1)}{1.} \frac{(m-2)}{2.} \frac{(m-3)}{3} Z^3 + \&c.$

73. Sit igitur primum $Z = \alpha z$, eritque $(1 + \alpha z)^{m-1} = 1 + \frac{m-1}{1} \alpha z + \frac{(m-1)}{1.} \frac{(m-2)}{2} \alpha^2 z^2 + \frac{(m-1)}{1.} \frac{(m-2)}{2.} \frac{(m-3)}{3} \alpha^3 z^3 + \&c.$ Scribatur pro hac Serie iſta forma generalis

$$1 + Az + Bz^2 + Cz^3 + \ldots\ldots\ldots + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens N ex præcedente M ita determinabitur ut ſit $N = \frac{m-n}{n} \alpha M$. Sic, poſito $n = 1$, cum ſit $M = 1$, erit $N = A = \frac{m-1}{1} \alpha$; tum facto $n = 2$, ob $M = A = \frac{m-1}{1} \alpha$, erit $N = B = \frac{m-2}{2} \alpha M = \frac{(m-1)}{1.} \frac{(m-2)}{2} \alpha^2$; ſimilique modo porro $C = \frac{m-3}{3} \alpha B = \frac{(m-1)}{1.} \frac{(m-2)}{2.} \frac{(m-3)}{3} \alpha^3$, uti Series ante inventa declarat.

74. Sit $Z = \alpha z + \beta zz$, eritque $(1 + \alpha z + \beta zz)^{m-1} = 1 + \frac{(m-1)}{1} (\alpha z + \beta zz) + \frac{(m-1)(m-2)}{1.\ 2} (\alpha z + \beta zz)^2 + \&c.$ Quod ſi ergo termini ſecundum poteſtates ipſius z diſponantur, erit $(1 + \alpha z + \beta zz)^{m-1} =$

$$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1.\ 2.} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3 z^3 + \&c.$$
$$+ \frac{(m-1)}{1} \beta z^2 + \frac{(m-1)(m-2)}{1.\ 2} 2\alpha\beta z^3 + \&c.$$

LIB. I. Scribatur pro hac Serie ista forma generalis:

$$1 + Az + Bz^2 + Cz^3 + \ldots + Lz^{n-2} + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens ex duobus antecedentibus ita definietur ut sit $N = \frac{m-n}{n}\alpha M + \frac{2m-n}{n}\beta L$, unde omnes termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$A = \frac{m-1}{1}\alpha;$$

$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$

$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A$$

$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B$$

&c.

75. Si fuerit $Z = \alpha z + \beta z^2 + \gamma z^3$, erit $(1 + \alpha z + \beta z^2 + \gamma z^3)^{m-1} = 1 + \frac{(m-1)}{1}(\alpha z + \beta z^2 + \gamma z^3) + \frac{(m-1)(m-2)}{1. \quad 2}(\alpha z + \beta z^2 + \gamma z^3)^2 + \&c.$, quæ expressio, si omnes termini secundum potestates ipsius z ordinentur, abibit in hanc Seriem:

$$1 + \frac{(m-1)}{1}\alpha z + \frac{(m-1)(m-2)}{1. \quad 2}\alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1. \quad 2. \quad 3}\alpha^3 z^3$$
$$+ \frac{(m-1)}{1}\beta z^2 + \frac{(m-1)(m-2)}{1. \quad 2}2\alpha\beta z^3 + \&c.$$
$$+ \frac{(m-1)}{1}\gamma z^3$$

cujus lex progressionis ut melius patescat, ponatur ejus loco $1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n$, cujus Seriei quilibet coëfficiens ex tribus antecedentibus ita determinatur ut sit $N = \frac{(m-n)}{n}\alpha M + \frac{(2m-n)}{n}\beta L + \frac{(3m-n)}{n}\gamma K$.

Cum igitur primus terminus fit $= 1$, & antecedentes nulli, erit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B$$
&c.

76. Generaliter ergo fi ponatur $(1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \&c.)^{m-1} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \&c.$, hujus Seriei finguli termini ita ex præcedentibus definientur, ut fit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A + \frac{(4m-4)}{4}\delta$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B + \frac{(4m-5)}{5}\delta A + \frac{(5m-5)}{5}\epsilon$$
&c.

quilibet fcilicet terminus per tot præcedentes determinatur, quot habentur litteræ α, β, γ, δ, &c. in Functione ipfius z cujus poteftas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam fupra §. 68. ubi fimilem formam $(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)^{-m-1}$ in Seriem infini-

 tam resolvimus; si enim loco m scribatur $-m$ atque litteræ α, β, γ, δ, &c. negative accipiantur, Series inventæ prorsus congruent. Interim hoc loco non licet rationem hujus progressionis legis a priori demonstrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.

CAPUT V.

De Functionibus duarum pluriumve variabilium.

77. QUANQUAM plures hactenus quantitates variabiles fumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius essent Functiones, unaque determinata reliquæ simul determinarentur. Nunc autem ejusmodi considerabimus quantitates variabiles, quæ a se invicem non pendeant, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejusmodi ergo quantitates variabiles, cujusmodi sint x, y, z, ratione significationis convenient, cum quælibet omnes valores determinatos in se complectatur; at, si inter se comparentur maxime erunt diversæ, cum, licet pro una z valor quicunque determinatus substituatur, reliquæ tamen x & y æque late pateant, atque ante. Discrimen ergo inter quantitates variabiles a se pendentes, & non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquæ determinentur; posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium*, x, y, z, *est expressio quomodocunque ex his quantitatibus composita.*

Ita erit $x^3 + xyz + az^2$ Functio quantitatum variabilium trium x, y, z. Hæc ergo Functio, si una determinetur va-

riabilis, puta z, hoc est ejus loco constans numerus substituatur, manebit adhuc quantitas variabilis, scilicet Functio ipsarum x & y. Atque si, præter z, quoque y determinetur, tum erit adhuc Functio ipsius x. Hujusmodi ergo plurium variabilium Functio non ante valorem determinatum obtinebit, quam singulæ quantitates variabiles fuerint determinatæ. Cum igitur una quantitas variabilis infinitis modis determinari possit, Functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes suscipere potest, omnino infinities infinitas determinationes admittet. Atque in Functione trium variabilium numerus determinationum erit adhuc infinities major; sicque porro crescet pro pluribus variabilibus. CAP. V.

79. *Hujusmodi Functiones plurium variabilium perinde atque Functiones unius variabilis, commodissime dividuntur in algebraicas ac transcendentes.*

Quarum illæ sunt, in quibus ratio compositionis in solis Algebræ operationibus est posita; hæ vero, in quarum formationem quoque operationes transcendentes ingrediuntur. In his denuo species notari possent, prout operationes transcendentes vel omnes quantitates variabiles implicant, vel aliquot, vel tantum unicam. Sic ista expressio $zz + y \log. z$, quia Logarithmus ipsius z inest, erit quidem Functio transcendens ipsarum y & z, verum ideo minus transcendens est putanda, quod si variabilis z determinetur, supersit Functio algebraica ipsius y. Interim tamen non expedit hujusmodi subdivisionibus tractationem amplificari.

80. *Functiones deinde algebraicæ subdividuntur in rationales & irrationales; rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex Capite primo jam abunde intelligitur. Functio scilicet rationalis omnino est libera ab omni irrationalitate quantitates variabiles, quarum Functio dicitur, afficiente; hæcque erit integra si nullis fractionibus inquinetur, contra vero fracta. Sic functionis integræ duarum variabilium y & z hæc erit forma generalis: $\alpha + \beta y + \gamma z + \delta y^2 + \epsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota y z^2 + \kappa z^3 +$ &c. Quod

 si ergo P & Q denotent hujusmodi Functiones integras, sive duarum sive plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis est vel explicita, vel implicita; illa per signa radicalia jam penitus est evoluta, hæc autem per æquationem irresolubilem exhibetur: sic V erit Functio implicita irrationalis ipsarum y & z, si fuerit $V^3 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili constant.*

Sic Functiones rationales erunt uniformes, quia singulis quantitatibus variabilibus determinatis, unicum valorem determinatum exhibent. Denotent P, Q, R, S, &c. Functiones rationales seu uniformes variabilium x, y, z, eritque V Functio biformis earundem variabilium, si fuerit $V^2 - PV + Q = 0$; quicunque enim valores determinati quantitatibus x, y, & z tribuantur, Functio V non unum sed duplicem perpetuo habebit valorem determinatum. Simili modo erit V Functio triformis si fuerit $V^3 - PV^2 + QV - R = 0$: atque Functio quadriformis si fuerit $V^4 - PV^3 + QV^2 - RV + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum si Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem consequitur determinatum vel simplicem vel multiplicem; ita si Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejusque ideo Functio evadit, cum ante a se mutuo non penderent. Simili modo si Functio trium variabilium x, y, z, nihilo æqualis statuatur, tum una variabilis per duas reliquas definitur, earumque Functio existit. Idem evenit si functio non nihilo sed quantitati constanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, semper una variabilis per reliquas definitur earumque fit Functio; duæ autem æqua-

tiones diversæ inter easdem variabiles ortæ binas per reliquas definient, atque ita porro. CAP. V.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas & heterogeneas.*

Functio homogenea est per quam ubique idem regnat variabilium numerus dimensionum: Functio autem heterogenea est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquæque variabilis unam dimensionem constituere; quadratum uniuscujusque atque productum ex duabus, duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres & ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis αy; βz, unica dimensio inesse dicitur; in his vero αy^2; βyz; γz^2 duæ insunt dimensiones: in his αy^3; $\beta y^2 z$; $\gamma y z^2$; δz^3, tres; in his vero αy^4; $\beta y^3 z$; $\gamma y^2 z^2$; $\delta y z^3$; ϵz^4; quatuor, sicque porro.

84. Applicemus primum hanc distinctionem ad Functiones integras, atque duas tantum variabiles inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea in cujus singulis terminis idem existit dimensionum numerus.

Subdividentur ergo hujusmodi Functiones commodissime secundum numerum dimensionum, quem variabiles in ipsis ubique constituunt. Sic erit $\alpha y + \beta z$ forma generalis Functionum integrarum unius dimensionis: hæc vero expressio $\alpha y^2 + \beta yz + \gamma z^2$ erit forma generalis Functionum duarum dimensionum, tum forma generalis Functionum trium dimensionum erit: $\alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3$; quatuor dimensionum vero hæc: $\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta y z^3 + \epsilon z^4$; & ita porro. Ad analogiam igitur erit quantitas constans sola α Functio nullius dimensionis.

85. *Functio porro fracta erit homogenea, si ejus Numerator ac Denominator fuerint Functiones homogeneæ.*

Sic hæc Fractio $\frac{\alpha yy + \beta zz}{\alpha y + \beta z}$ erit Functio homogenea ipsa-

 rum y & z; numerus dimenſionum autem habebitur, ſi a numero dimenſionum Numeratoris ſubtrahatur numerus dimenſionum Denominatoris: atque ob hanc rationem Fractio allata erit Functio unius dimenſionis. Hæc vero Fractio $\frac{y^5 + z^5}{yy + zz}$ erit Functio trium dimenſionum. Quando ergo in Numeratore ac Denominatore idem dimenſionum numerus ineſt, tum Fractio erit Functio nullius dimenſionis, uti evenit in hac Fractione $\frac{y^3 + z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{\alpha zz}{yy}$; $\frac{\beta y^3}{z^3}$. Quod ſi igitur in Denominatore plures ſint dimenſiones quam in Numeratore, numerus dimenſionum Fractionis erit negativus: ſic $\frac{y}{zz}$ erit Functio -1 dimenſionis: $\frac{y + z}{y^4 + z^4}$ erit Functio -3 dimenſionum: $\frac{1}{y^5 + ayz^4}$ erit Functio -5 dimenſionum, quia in Numeratore nulla ineſt dimenſio. Ceterum ſponte intelligitur plures Functiones homogeneas, in quibus ſingulis idem regnat dimenſionum numerus, ſive additas ſive ſubtractas præbere Functionem quoque homogeneam ejuſdem dimenſionum numeri. Sic hæc expreſſio $\alpha y + \frac{\beta zz}{y} + \frac{\gamma y^4 - \delta z^4}{yyz + yzz}$ erit Functio unius dimenſionis: hæc autem $\alpha + \frac{\beta y}{z} + \frac{\gamma zz}{yy} + \frac{yy + zz}{yy - zz}$ erit Functio nullius dimenſionis.

86. Natura Functionum homogenearum quoque ad expreſſiones irrationales extenditur. Si enim fuerit P Functio quæcunque homogenea, puta n dimenſionum, tum $\sqrt{P}$ erit Functio $\frac{1}{2}.n$ dimenſionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3}n$ dimenſionum, & generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu}n$ dimenſionum. Sic $\sqrt{(yy+zz)}$ erit Functio unius dimenſionis; $\sqrt[3]{(y^9 + z^9)}$ erit Functio trium dimenſionum: $(yz + zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimenſionum:

atque

atque $\frac{yy + zz}{\sqrt{(y^4 + z^4)}}$ erit Functio nullius dimensionis. His ergo cum præcedentibus conjunctis intelligitur hæc expressio $\frac{1}{y} + \frac{y\sqrt{(yy+zz)}}{z^3} - \frac{y}{\sqrt[3]{(y^6 - z^6)}} + \frac{y\sqrt{z}}{zz\sqrt{y} + \sqrt{(y^5 + z^5)}}$ esse Functio homogenea -1 dimensionis.

87. Utrum Functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit V hujusmodi Functio implicita ac $V^3 + PV^2 + QV + R = 0$, existentibus P, Q & R Functionibus ipsarum y & z. Primum igitur patet V Functionem homogeneam esse non posse, nisi P, Q, & R sint Functiones homogeneæ. Præterea vero si ponamus V esse Functionem n dimensionum, erit V^2 Functio $2n$, & V^3 Functio $3n$ dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut P sit Functio n dimensionum, Q Functio $2n$ dimensionum, & R Functio $3n$ dimensionum. Sint ergo vicissim litteræ P, Q, R Functiones homogeneæ respective n, $2n$, $3n$ dimensionum, hinc concludetur fore V Functionem n dimensionum. Ita si fuerit $V^5 + (y^4 + z^4)V^3 + ay^8V - z^{10} = 0$ erit V Functio homogenea duarum dimensionum, ipsarum y & z.

88. *Si fuerit* V *Functio homogenea* n *dimensionum ipsarum* y *&* z, *in eaque ponatur ubique* $y = uz$, *Functio* V *abibit in productum ex potestate* z^n *in Functionem quandam variabilis* u.

Per hanc enim substitutionem $y = uz$, in singulos terminos tantæ inducentur potestates ipsius z, quantæ ante inerant ipsius y. Cum igitur in singulis terminis dimensiones ipsarum y & z conjunctim æquassent numerum n, nunc sola variabilis z ubique habebit n dimensiones, ideoque ubique inerit ejus potestas z^n. Per hanc ergo potestatem Functio V fiet divisibilis & quotus erit Functio variabilem tantum u involvens. Hoc primum patebit in Functionibus integris; si enim sit $V = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3$, posito $y = uz$, fiet $V = z^3$

LIB. I. $(\alpha u^3 + \beta u^2 + \gamma u + \delta)$. Deinde vero idem manifeſtum eſt in fractis: ſit enim $V = \frac{\alpha y + \beta z}{yy + zz}$, nempe Functio -1 dimenſionis, facto $y = uz$ fiet $V = z^{-1} (\frac{\alpha u + \beta}{uu + 1})$. Neque etiam Functiones irrationales hinc excipiuntur, ſi enim ſit $V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}}$, quæ eſt Functio $-\frac{3}{2}$ dimenſionum; poſito $y = uz$, prodibit $V = z^{-\frac{3}{2}} (\frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}})$. Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim poteſtas ipſius z, quia eſt Factor, Functionem illam ipſius u inquinat.

89. *Functio ergo homogenea* V *duarum variabilium* y *&* z *nullius dimenſionis, poſito* y = u z, *tranſmutabitur in Functionem unicæ variabilis* u *puram.*

Cum enim numerus dimenſionum ſit nullus, Poteſtas ipſius z, quæ Functionem ipſius u multiplicabit, erit $z^0 = 1$; hocque caſu variabilis z prorſus ex computo egredietur. Ita ſi fuerit $V = \frac{y + z}{y - z}$, facto $y = uz$, orietur $V = \frac{u + 1}{u - 1}$: atque in irrationalibus ſi ſit $V = \frac{y - \sqrt{(yy - zz)}}{z}$ poſito $y = uz$ erit $V = u - \sqrt{(uu - 1)}$.

90. *Functio integra homogenea duarum variabilium* y *&* z, *reſolvi poterit in tot Factores ſimplices formæ* α y + β z, *quot habuerit dimenſiones.*

Cum enim Functio ſit homogenea, poſito $y = uz$, tranſibit in productum ex z^n in Functionem quandam ipſius u integram, quæ Functio propterea in Factores ſimplices formæ $\alpha u + \beta$ reſolvi poterit. Multiplicentur ſinguli Factores hi per z, eritque uniuſcujuſque forma $\alpha uz + \beta z = \alpha y + \beta z$ ob $uz = y$. Propter multiplicatorem autem z^n, tot hujuſmodi Factores naſcentur quot exponens n contineat unitates; Factores autem

hi ſimplices erunt vel reales vel imaginarii; hoc eſt, coëfficientes α & β erunt vel reales vel imaginarii. CAP. V.

Ex hoc itaque ſequitur Functionem duarum dimenſionum $ayy + byz + czz$ duos habere Factores ſimplices formæ $\alpha y + \beta z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres Factores ſimplices formæ $\alpha y + \beta z$; ſicque porro Functionum homogenearum integrarum, quæ plures habent dimenſiones, natura erit comparata.

91. Quemadmodum ergo hæc expreſſio $\alpha y + \beta z$ continet formam generalem Functionum integrarum unius dimenſionis, ita $(\alpha y + \beta z)(\gamma y + \delta z)$ erit forma generalis Functionum integrarum duarum dimenſionum: atque in hac forma $(\alpha y + \beta z)(\gamma y + \delta z)(\epsilon y + \zeta z)$ continebuntur omnes Functiones integræ trium dimenſionum, ſicque omnes Functiones integræ homogeneæ per producta ex tot hujuſmodi Factoribus $\alpha y + \beta z$ exhiberi poterunt, quot Functiones illæ contineant dimenſiones. Iſti autem Factores eodem modo per reſolutionem æquationum reperiuntur, quo ſupra Factores ſimplices Functionum integrarum unius variabilis invenire docuimus. Ceterum hæc proprietas Functionum homogenearum duarum variabilium non extenditur ad Functiones homogeneas trium pluriumve variabilium: forma enim generalis hujuſmodi Functionum duarum tantum dimenſionum, quæ eſt $ayy + byz + cyx + dxy + exx + fzz$ generaliter non reduci poteſt ad hujuſmodi productum $(\alpha y + \beta z + \gamma x)(\delta y + \epsilon z + \zeta x)$; multoque minus Functiones plurium dimenſionum ad hujuſmodi producta revocari poſſunt.

92. Ex his, quæ de Functionibus homogeneis ſunt dicta, ſimul intelligitur, quid ſit Functio heterogenea: in cujus ſcilicet terminis non ubique idem dimenſionum numerus deprehenditur. Poſſunt autem Functiones heterogeneæ ſubdividi pro multiplicitate dimenſionum, quæ in ipſis occurrunt. Sic Functio bifida erit, in qua duplex dimenſionum numerus occurrit, eritque adeo aggregatum duarum Functionum homo-

 genearum, quarum numeri dimensionum differunt; ita $y^5 + 2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem trifida est, in qua tres diversi dimensionum numeri insunt, seu quæ in tres Functiones homogeneas distribui potest, uti $y^6 + y^2z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas resolvi non possunt, cujusmodi sunt $\frac{y^3 + ayz}{by + zz}$, $\frac{a + \sqrt{(yy + zz)}}{yy - bz}$.

93. Interdum Functio heterogenea ope substitutionis idoneæ, vel loco unius vel utriusque variabilis factæ, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attulisse, quibus ejusmodi reductio locum habet. Si scilicet hæc proposita sit Functio $y^5 + zzy + y^3z + \frac{z^3}{y}$; post levem attentionem apparebit, eam ad homogeneitatem perduci, posito $z = xx$: prodibit enim $y^5 + x^4y + y^3xx + \frac{x^6}{y}$, Functio homogenea 5 dimensionum ipsarum x & y. Deinde hæc Functio $y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$, prodit enim Functio unius dimensionis $y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az$. Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero qui in Functione inest. Sic $xx + yy + zz + ay - aa$ est Functio secundi ordinis, quia duæ dimensiones occurrunt. Et $y^4 + yz^3 - ay^2z +$

$abyz - aayy + b^4$ pertinet ad Functiones quarti ordinis. Ad hanc divisionem potissimum in doctrina de lineis curvis respici solet; unde adhuc una Functionum integrarum divisio commemoranda venit.

95. Superest scilicet divisio Functionum integrarum in complexas atque incomplexas. Functio autem complexa est, quæ in Factores rationales resolvi potest, seu quæ est productum ex duabus Pluribusve functionibus rationalibus; cujusmodi est $y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy$, quæ est productum ex his duabus Functionibus $(yy + zz - az + by)$ $(yy - zz + az - by)$. Ita vidimus omnem Functionem integram homogeneam, quæ tantum duas variabiles complectatur, esse Functiouem complexam, quoniam tot Factores simplices formæ $\alpha y + \beta z$ habet, quot continet dimensiones. Functio igitur integra erit incomplexa, si in Factores rationales resolvi omnino nequeat; uti $yy + zz - aa$, cujus nullos dari Factores rationales facile intelligitur. Ex inquisitione Divisorum patebit, utrum Functio proposita sit complexa an incomplexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. QUANQUAM notio Functionum transcendentium in calculo integrali demum perpendetur, tamen antequam eo perveniamus, quasdam species magis obvias, atque ad plures investigationes aditum aperientes, evolvere conveniet. Primum ergo considerandæ sunt quantitates exponentiales, seu Potestates, quarum Exponens ipse est quantitas variabilis. Perspicuum enim est hujusmodi quantitates ad Functiones algebraicas referri non posse, cum in his Exponentes non nisi constantes locum habeant. Multiplices autem sunt quantitates exponen-

LIB. I. tiales, prout vel ſolus Exponens eſt quantitas variabilis, vel præterea etiam ipſa quantitas elevata; prioris generis eſt a^z, hujus vero y^z; quin etiam ipſe Exponens poteſt eſſe quantitas exponentialis uti in his formis a^{a^z}; a^{y^z}; y^{a^z}; x^{y^z}. Hujuſmodi autem quantitatum non plura conſtituemus genera, cum earum natura ſatis clarè intelligi queat, ſi primam tantum ſpeciem a^z evolverimus.

97. Sit igitur propoſita hujuſmodi quantitas exponentialis a^z, quæ eſt Poteſtas quantitatis conſtantis a, Exponentem habens variabilem z. Cum igitur iſte Exponens z omnes numeros determinatos in ſe complectatur, primum patet ſi loco z omnes numeri integri affirmativi ſucceſſive ſubſtituantur, loco a^z hos prodituros eſſe valores determinatos a^1; a^2; a^3; a^4; a^5; a^6; &c. Sin autem pro z ponantur ſucceſſive numeri negativi -1, -2, -3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c. ac, ſi fuerit $z = 0$, habebitur ſemper $a^0 = 1$. Quod ſi loco z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c. orientur iſti valores $\sqrt{a}$; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in ſe ſpectati geminos pluresve induunt valores, cum radicum extractio ſemper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales ſcilicet atque affirmativi admitti ſolent; quia quantitas a^z tanquam Functio uniformis ipſius z ſpectatur. Sic $a^{\frac{5}{2}}$ medium quemdam tenebit locum inter a^2 & a^3, eritque ideo quantitas ejuſdem generis; & quamvis valor $a^{\frac{5}{2}}$ ſit æque $= -aa\sqrt{a}$, ac $= +aa\sqrt{a}$, tamen poſterior tantum in cenſum venit. Eodem modo res ſe habet, ſi Exponens z valores irrationales accipiat, quibus caſibus cum difficile ſit numerum valorum involutorum conci-

pere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor CAP. VI.
determinatus intra limites a^2 & a^3 comprehensus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis a pendebunt. Si enim fuerit $a = 1$, semper erit $a^z = 1$, quicunque valores Exponenti z tribuantur; sin autem fuerit $a > 1$, tum valor ipsius a^z eo erit major, quo major numerus loco z substituatur, atque adeo, posito $z = \infty$, a^z in infinitum excrescit; si fuerit $z = 0$, fiet $a^z = 1$, &, si sit $z < 0$ valores a^z fient unitate minores, quoad posito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit si sit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent, crescente z supra 0; crescent vero, si pro z numeri negativi substituantur. Cum enim sit $a < 1$, erit $\frac{1}{a} > 1$; posito ergo $\frac{1}{a} = b$; erit $a^z = b^{-z}$, unde posterior casus ex priori dijudicari poterit.

99. Si sit $a = 0$, ingens saltus in valoribus ipsius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus seu major nihilo, erit perpetuo $a^z = 0$: si sit $z = 0$ erit $a^0 = 1$; sin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, ideoque infinitum. Multo majores autem saltus occurrent, si quantitas constans a habeat valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4};\ a^{-3};\ a^{-2};\ a^{-1};\ a^0;\ a^1;\ a^2;\ a^3;\ a^4;\ \&c.$$

LIB. I. $+\frac{1}{16}; -\frac{1}{8}; +\frac{1}{4}; -\frac{1}{2}; 1; -2; +4; -8; +16.$ Præterea vero ſi Exponenti z valores tribuantur fracti, Poteſtas $a^z = (-2)^z$ mox reales mox imaginarios induet valores: erit enim $a^{\frac{1}{2}} = \sqrt{-2}$ imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, ſi Exponenti z tribuantur valores irrationales, Poteſtas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a ſubſtituendorum commemoratis, ſtatuamus a eſſe numerum affirmativum, & unitate quidem majorem, quia huc quoque illi caſus, quibus a eſt numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z ſubſtituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiſcetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim ſit $z = \infty$ erit $y = \infty$; ſi $z = 0$ erit $y = 1$, & ſi $z = -\infty$ fiet $y = 0$. Viciſſim ergo quicunque valor affirmativus pro y accipiatur; dabitur quoque valor realis reſpondens pro z ita ut ſit $a^z = y$; ſin autem ipſi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipſius z, &, quemadmodum y a z pendeat, ex natura Poteſtatum facile intelligitur; hinc enim quicunque valor ipſi z tribuatur, valor ipſius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$: & generaliter erit $y^n = a^{nz}$; unde ſequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, ſi fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum ſubſidiorum beneficio eo facilius valor ipſius y ex dato valore ipſius z inveniri poteſt.

EXEMPLUM,

Exemplum.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promtu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: sin autem pro z Fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt: sic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$; &c.

102. Quemadmodum autem, dato numero a, ex quovis valore ipsius z reperiri potest valor ipsius y, ita vicissim, dato valore quocunque affirmativo ipsius y, conveniens dabitur valor ipsius z, ut sit $a^z = y$; iste autem valor ipsius z, quatenus tanquam Functio ipsius y spectatur, vocari solet Logarithmus ipsius y. Supponit ergo doctrina Logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* Logarithmorum; qua assumta erit Logarithmus cujusque numeri y Exponens Potestatis a^z, ita ut ipsa Potestas a^z aequalis sit numero illi y; indicari autem Logarithmus numeri y solet hoc modo ly. Quod si ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, basin Logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate majorem: hincque nonnisi numerorum affirmativorum Logarithmos realiter exhiberi posse.

103. Quicunque ergo numerus pro basi Logarithmica a accipiatur, erit semper $l1 = 0$; si enim in aequatione $a^z = y$, quae convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore basis a, sic erit $la = 1$; $laa = 2$; $la^3 = 3$;

LIB. I. $la^4 = 4$; &c., unde a posteriori intelligi potest, quantus numerus pro basi sit assumtus, scilicet ille numerus, cujus Logarithmus est $= 1$, erit basis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l\frac{1}{a} = -1$; $l\frac{1}{aa} = -2$; $l\frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, sed imaginarii, uti jam notavimus.

104. Simili modo si fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $ly^n = nz$, seu $ly^n = nly$, ob $z = ly$. Logarithmus igitur cujusque Potestatis ipsius y æquatur Logarithmo ipsius y per Exponentem Potestatis multiplicato; sic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro: unde ex dato Logarithmo cujusque numeri inveniri possunt Logarithmi quarumcunque ipsius Potestatum. Sin autem jam inventi sint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum sit $y = a^z$ & $v = a^x$ erit $lvy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur summæ Logarithmorum Factorum; simili vero modo erit $l\frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris: quæ regulæ inserviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, nisi Potestatum baseos a; nisi enim numerus alius b fuerit Potestas basis a, ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipsius b erit numerus irrationalis; si enim foret $lb = \sqrt{n}$, tum esset $a^{\sqrt{n}} = b$; id quod fieri nequit, si quidem numeri a & b

rationales statuantur; solent autem imprimis numerorum rationalium & integrorum Logarithmi desiderari, quia ex his Logarithmi Fractionum ac numerorum surdorum inveniri possunt. Cum igitur Logarithmi numerorum, qui non sunt Potestates basis a, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates transcendentes referuntur, hincque Logarithmi quantitatibus transcendentibus annumerari solent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi solent, qui eo minus à veritate discrepabunt, quod ad plures figuras fuerint exacti. Atque hoc modo per solam radicis quadratæ extractionem cujusque numeri Logarithmus vero proxime determinari poterit. Cum enim, posito $ly = z$ & $lv = x$, sit $l\sqrt{vy} = \frac{x+z}{2}$; si numerus propositus b contineatur intra limites a^2 & a^3, quorum Logarithmi sunt 2 & 3, quæratur valor ipsius $a^{2\frac{1}{2}}$ seu $a^2\sqrt{a}$, atque b vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accidat, sumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibuscum numerus propositus b sine errore confundi possit. Quoniam vero horum singulorum limitum Logarithmi dantur, tandem Logarithmus numeri b reperietur.

EXEMPLUM.

Ponatur basis Logarithmica $a = 10$, quod in tabulis usu receptis fieri solet; & quæratur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi sunt 0 & 1; sequenti modo radicum extractio continua instituatur, quoad ad limites à numero proposito 5 non amplius discrepantes perveniatur.

LIB. I.

$A = 1,\ 000000;$	$lA = 0,\ 0000000;$	fit
$B = 10,\ 000000;$	$lB = 1,\ 0000000;$	$C = \sqrt{AB}$
$C = 3,\ 162277;$	$lC = 0,\ 5000000;$	$D = \sqrt{BC}$
$D = 5,\ 623413;$	$lD = 0,\ 7500000;$	$E = \sqrt{CD}$
$E =$ [illegible]$,\ 216964;$	$lE = 0,\ 6250000;$	$F = \sqrt{DE}$
$F =$ [illegible]$,\ 869674;$	$lF = 0,\ 6875000;$	$G = \sqrt{DF}$
$G = 5,\ 232991;$	$lG = 0,\ 7187500;$	$H = \sqrt{FG}$
$H = 5,\ 048065;$	$lH = 0,\ 7031250;$	$I = \sqrt{FH}$
$I = 4,\ 958069;$	$lI = 0,\ 6953125;$	$K = \sqrt{HI}$
$K = 5,\ 002865;$	$lK = 0,\ 6992187;$	$L = \sqrt{IK}$
$L = 4,\ 980416;$	$lL = 0,\ 6972656;$	$M = \sqrt{KL}$
$M = 4,\ 991627;$	$lM = 0,\ 6982421;$	$N = \sqrt{KM}$
$N = 4,\ 997242;$	$lN = 0,\ 6987304;$	$O = \sqrt{KN}$
$O = 5,\ 000052;$	$lO = 0,\ 6989745;$	$P = \sqrt{NO}$
$P = 4,\ 998647;$	$lP = 0,\ 6988525;$	$Q = \sqrt{OP}$
$Q = 4,\ 999350;$	$lQ = 0,\ 6989135;$	$R = \sqrt{OQ}$
$R = 4,\ 999701;$	$lR = 0,\ 6989440;$	$S = \sqrt{OR}$
$S = 4,\ 999876;$	$lS = 0,\ 6989592;$	$T = \sqrt{OS}$
$T = 4,\ 999963;$	$lT = 0,\ 6989668;$	$V = \sqrt{OT}$
$V = 5,\ 000008;$	$lV = 0,\ 6989707;$	$W = \sqrt{TV}$
$W = 4,\ 999984;$	$lW = 0,\ 6989687;$	$X = \sqrt{WV}$
$X = 4,\ 999997;$	$lX = 0,\ 6989697;$	$Y = \sqrt{VX}$
$Y = 5,\ 000003;$	$lY = 0,\ 6989702;$	$Z = \sqrt{XY}$
$Z = 5,\ 000000;$	$lZ = 0,\ 6989700;$	

Sic ergo mediis proportionalibus ſumendis tandem perventum eſt ad $Z = 5,\ 000000$, ex quo Logarithmus numeri 5 quæſitus eſt $0,698970$, poſita baſi Logarithmica $= 10$. Quare erit proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus eſt canon Logarithmorum vulgaris à BRIGGIO & VLACQUIO, quamquam poſtea eximia inventa ſunt compendia, quorum ope multo expeditius Logarithmi ſupputari poſſunt.

107. Dantur ergo tot diverſa Logarithmorum ſyſtemata quot varii numeri pro baſi a accipi poſſunt, atque ideo numerus ſyſ-

tematum Logarithmicorum erit infinitus. Perpetuo autem in duobus systematis Logarithmi ejusdem numeri eandem inter se servant rationem. Sit basis unius systematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori systemate $= p$, in posteriori $= q$; erit $a^p = n$ & $b^q = n$; unde $a^p = b^q$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ constantem obtineat valorem, quicunque numerus pro n fuerit assumtus. Quod si ergo pro uno systemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio systemate reperiri possunt. Sic, cum dentur Logarithmi pro basi 10, hinc Logarithmi pro quavis alia basi, puta 2, inveniri possunt; quæratur enim Logarithmus numeri n pro basi 2, qui sit $= q$, cum ejusdem numeri n Logarithmus sit $= p$ pro basi 10. Quoniam pro basi 10 est $l\,2 = 0,3010300$, & pro basi 2, est $l\,2 = 1$, erit $0,3010300 : 1 = p : q$ ideoque $q = \frac{p}{0,3010300} = 3,3219277.\,p$; si ergo omnes Logarithmi communes multiplicentur per numerum 3, 3219277, prodibit tabula Logarithmorum pro basi 2.

108. *Hinc sequitur duorum numerorum Logarithmos in quocunque systemate eandem tenere rationem.*

Sint enim duo numeri M & N, quorum pro basi a Logarithmi sint m & n, erit $M = a^m$ & $N = a^n$: hinc fiet $a^{mn} = M^n = N^m$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatione cum basis a non amplius insit, perspicuum est Fractionem $\frac{m}{n}$ habere valorem à basi a non pendentem. Sint enim pro alia basi b numerorum eorundem M & N Logarithmi μ & ν, pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, seu $m : n = \mu : \nu$. Ita jam vidimus

 in omni Logarithmorum ſyſtemate Logarithmos diverſarum ejuſdem numeri Poteſtatum ut y^m & y^n tenere rationem Exponentium $m : n$.

109. Ad canonem ergo Logarithmorum pro baſi quacunque a condendum ſufficit numerorum tantum primorum Logarithmos methodo ante tradita, vel alia commodiori, ſupputaſſe. Cum enim Logarithmi numerorum compoſitorum ſint æquales ſummis Logarithmorum ſingulorum Factorum, Logarithmi numerorum compoſitorum per ſolam additionem reperientur. Sic, ſi habeantur logarithmi numerorum 3 & 5, erit $l\,15 = l\,3 + l\,5$; $l\,45 = 2\,l\,3 + l\,5$. Atque, cum ſupra pro baſi $a = 10$, inventus ſit $l\,5 = 0,6989700$, præterea autem ſit $l\,10 = 1$, erit $l\,\frac{10}{5} = l\,2 = l\,10 - l\,5$, ideoque orietur $l\,2 = 1 - 0,6989700 = 0,3010300$; ex his autem numerorum primorum 2 & 5 Logarithmis inventis reperientur Logarithmi omnium numerorum ex his 2 & 5 compoſitorum; cujuſmodi ſunt iſti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50, &c.

110. Tabularum autem Logarithmicarum ampliſſimus eſt uſus in contrahendis calculis numericis, propterea quod ex ejuſmodi tabulis non ſolum dati cujuſque numeri Logarithmus, ſed etiam cujuſque Logarithmi propoſiti numerus conveniens reperiri poteſt. Sic, ſi c, d, e, f, g, h, denotent numeros quoſcunque, citra multiplicationem reperiri poterit valor iſtius expreſſionis $\frac{cc\,d\,\sqrt{e}}{f\,\sqrt[3]{g\,h}}$, erit enim hujus expreſſionis Logarithmus $= 2\,l\,c + l\,d + \frac{1}{2}\,l\,e - l\,f - \frac{1}{3}\,l\,g - \frac{1}{3}\,l\,h$, cui Logarithmo ſi quæratur numerus reſpondens, habebitur valor quæſitus. Inprimis autem inſerviunt tabulæ Logarithmicæ dignitatibus atque radicibus intricatiſſimis inveniendis, quarum operationum loco in Logarithmis tantum multiplicatio aut diviſio adhibetur.

EXEMPLUM I.

Quæratur valor hujus Poteſtatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus eſt $= \frac{7}{12}\,l\,2$, multiplicetur Logarithmus binarii ex tabulis qui eſt 0, 3010300 per $\frac{7}{12}$ hoc eſt per $\frac{1}{2} + \frac{1}{12}$ erit, $l\,2^{\frac{7}{12}} = 0,\ 1756008$, cui Logarithmo reſpondet numerus 1, 498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuſpiam provinciæ quotannis ſui parte trigeſima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur poſt 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut ſit $n =$ 100000, anno elapſo uno erit incolarum numerus $= (1 + \frac{1}{30})n = \frac{31}{30}n$: poſt duos annos $= (\frac{31}{30})^2 n$: poſt tres annos $= (\frac{31}{30})^3 n$, hincque poſt centum annos $= (\frac{31}{30})^{100} n =$ $(\frac{31}{30})^{100}\ 100000$; cujus Logarithmus eſt $= 100\ l\,\frac{31}{30} +$ $l\ 100000$. At eſt $l\,\frac{31}{30} = l\,31 - l\,30 = 0,\ 014240439$, unde $100\ l\,\frac{31}{30} = 1,\ 4240439$, ad quem ſi addatur $l\ 100000$ $= 5$, erit Logarithmus numeri incolarum quæſiti $= 6{,}4240439$, cui reſpondet numerus $= 2654874$. Poſt centum ergo annos numerus incolarum fit plus quam vicies ſexies cum ſemiſſe major.

EXEMPLUM III.

Cum post diluvium a sex hominibus genus humanum sit propagatum, si ponamus ducentis annis post, numerum hominum jam ad 1000000 excrevisse, quæritur quanta sui parte numerus hominum quotannis augeri debuerit. Ponamus hoc tempore numerum hominum parte sua $\frac{1}{x}$ quotannis increvisse, atque post ducentos annos prodierit necesse est numerus hominum $= \left(\frac{1+x}{x}\right)^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}}$. Erit ergo $l\,\frac{1+x}{x} = \frac{1}{200}\, l\, \frac{1000000}{6} = \frac{1}{200}.$ $5,2218487 = 0,0261092$, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, & $1000000 = 61963\,x$, unde fit $x = 16$ circiter. Ad tantam ergo hominum multiplicationem suffecisset, si quotannis decima sexta sui parte increvissent; quæ multiplicatio ob longævam vitam non nimis magna censeri potest. Quod si autem eadem ratione per intervallum 400 annorum numerus hominum crescere perrexisset, tum numerus hominum ad $1000000.\,\frac{1000000}{6} = 166666666666$ ascendere debuisset, quibus sustentandis universus orbis terrarum nequaquam par fuisset.

EXEMPLUM IV.

Si singulis seculis numerus hominum duplicetur, quæritur incrementum annuum. Si quotannis hominum numerum parte sua $\frac{1}{x}$ crescere ponamus, & initio numerus hominum fuerit $= n$, erit is post centum annos $= \left(\frac{1+x}{x}\right)^{100} n$, qui cum esse de-

beat

CAP. VI.

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l\frac{1+x}{x} = \frac{1}{100} l2 =$ $0,0030103$; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x =$ $\frac{10000000}{69555} = 144$, circiter; sufficit ergo si numerus hominum quotannis parte sua $\frac{1}{144}$ augeatur. Quam ob causam maxime ridiculæ sunt eorum incredulorum hominum objectiones, qui negant tam brevi temporis spatio ab uno homine universam terram incolis impleri potuisse.

111. Potissimum autem Logarithmorum usus requiritur ad ejusmodi æquationes resolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, si ad hujusmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non nisi per Logarithmos effici poterit. Cum enim sit $a^x = b$ erit $la^x = xla = lb$, ideoque $x = \frac{lb}{la}$, ubi quidem perinde est, quonam systemate Logarithmico utatur, cum in omni systemate Logarithmi numerorum a & b eandem inter se teneant rationem.

EXEMPLUM I.

Si numerus hominum quotannis centesima sui parte augeatur; quæritur post quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire post x annos, & initio hominum numerum fuisse $= n$, erit is ergo elapsis x annis $=$ $(\frac{101}{100})^x n$, qui cum æqualis sit $10n$, fiet $(\frac{101}{100})^x = 10$; ideoque $xl\frac{101}{100} = l10$ & $x = \frac{l10}{l101 - l100}$. Prodibit itaque $x = \frac{10000000}{43214} = 231$. Post annos ergo 231 fiet hominum

 numerus, quorum incrementum annuum tantum centesimam partem efficit, decuplo major; hinc post 462 annos fiet centies, & post 693 annos millies major.

Exemplum II.

Quidam debet 400000 florenos hac conditione ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos: quæritur post quot annos debitum penitus extinguatur. Scribamus a pro debita summa 400000 fl. & b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno $\frac{105}{100}\,a - b$; elapsis duobus annis $(\frac{105}{100})^2\,a - \frac{105}{100}\,b - b$; elapsis tribus annis $(\frac{105}{100})^3\,a - (\frac{105}{100})^2\,b - \frac{105}{100}\,b - b$; hinc, posito brevitatis causa, n pro $\frac{105}{100}$, elapsis x annis adhuc debebit $n^x\,a - n^{x-1}\,b - n^{x-2}\,b - n^{x-3}\,b - \ldots\ldots\ldots - b = n^x\,a - b\,(1 + n + n^2 + \ldots + n^{x-1})$. Cum igitur sit ex natura progressionum geometricarum, $1 + n + n^2 + \ldots\ldots\ldots + n^{x-1} = \frac{n^x - 1}{n - 1}$, post x annos debitor adhuc debebit $n^x a - \frac{n^x\,b + b}{n - 1}$ flor., quod debitum nihilo æquale positum dabit hanc æquationem $n^x\,a = \frac{n^x\,b - b}{n - 1}$, seu $(n - 1)\,n^x\,a = n^x\,b - b$, idcoque $(b - n\,a + a)\,n^x = b$ & $n^x = \frac{b}{b - (n - 1)\,a}$, unde fit $x = \frac{l\,b - l\,(b - (n - 1)\,a)}{l\,n}$. Cum jam sit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1)\,a = 20000$ & $b - (n - 1)\,a = 5000$, atque annorum, quibus debitum penitus extinguitur, numerus $x =$

$\frac{l\,25000 - l\,5000}{l\frac{105}{100}} = \frac{l\,5}{l\frac{21}{20}} = \frac{6989700}{211893}$; erit ergo x aliquanto minor quam 33; ſcilicet elapſis annis 33 non ſolum debitum extinguetur, ſed creditor debitori reddere tenebitur $\frac{(n^{33} - 1)b}{n - 1} - n^{33}a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000(\frac{21}{20})^{33} - 500000$ flor. Quia vero eſt $l\frac{21}{20} = 0,0211892991$, erit $l(\frac{21}{20})^{33} = 0,69924687$, & $l\,100000(\frac{21}{20})^{33} = 5,6992469$, cui reſpondet hic numerus $500318,8$; unde creditor debitori poſt 33 annos reſtituere debet $318\frac{4}{5}$ florenos. CAP. VI.

112. Logarithmi autem vulgares ſuper baſi $= 10$ inſtructi, præter hunc uſum, quem Logarithmi in genere præſtant, in Arithmetica decimali uſu recepta ſingulari gaudent commodo, atque ob hanc cauſam præ aliis ſyſtematibus inſignem afferunt utilitatem. Cum enim Logarithmi omnium numerorum, præter denarii Poteſtates, in Fractionibus decimalibus exhibeantur, numerorum inter 1 & 10 contentorum Logarithmi intra limites 0 & 1, numerorum autem inter 10 & 100 contentorum Logarithmi inter limites 1 & 2, & ita porro, continebuntur. Conſtat ergo Logarithmus quiſque ex numero integro & Fractione decimali; & ille numerus integer vocari ſolet CHARACTERISTICA; Fractio decimalis autem MANTISSA. Characteriſtica itaque unitate deficiet a numero notarum, quibus numerus conſtat; ita Logarithmi numeri 78509 Characteriſtica erit 4, quia is ex quinque notis ſeu figuris conſtat. Hinc ex Logarithmo cujuſvis numeri ſtatim intelligitur, ex quot figuris numerus ſit compoſitus. Sic numerus Logarithmo 7,5804631 reſpondens ex 8 figuris conſtabit.

113. Si ergo duorum Logarithmorum Mantiſſæ conveniant, Characteriſticæ vero tantum diſcrepent, tum numeri his Loga-

rithmis respondentes rationem habebunt, ut Potestas denarii ad unitatem, ideoque ratione figurarum, quibus constant, convenient. Ita horum Logarithmorum, 4, 9130187 & 6, 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3, 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantissa indicabit figuras numerum componentes, quibus inventis, ex Characteristica patebit, quot figuræ a sinistra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, si hic Logarithmus fuerit inventus 2, 7603429, Mantissa indicabit has figuras 5758945, Characteristica 2 autem numerum illi Logarithmo respondentem determinat, ut sit 575, 8945; si Characteristica esset 0, foret numerus 5, 758945; sin denuo unitate minuatur ut sit — 1, erit numerus respondens decies minor, nempe 0, 5758945; & Characteristicæ — 2 respondebit 0, 05758945 &c.: loco Characteristicarum autem hujusmodi negativarum — 1, — 2, — 3, &c. scribi solent 9, 8, 7, &c., atque subintelligitur hos Logarithmos denario minui debere. Hæc vero in manuductionibus ad tabulas Logarithmorum fusius exponi solent.

Exemplum.

Si hæc progressio 2, 4, 16, 256, &c., cujus quisque terminus est quadratum præcedentis, continuetur usque ad terminum vigesimum quintum; quæritur magnitudo hujus termini ultimi. Termini hujus progressionis per Exponentes ita commodius exprimuntur: $2^1, 2^2, 2^4, 2^8$, &c. ubi patet Exponentes progressionem geometricam constituere, atque termini vigesimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipse terminus quæsitus sit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216 . l2$. Cum ergo sit $l2 = 0, 30102999566398 1195$, erit numeri quæsiti Logarithmus $= 5050445, 25973367$, ex cujus Characteristica patet numerum quæsitum more solito expressum constare

ex 5050446 figuris. Mantiſſa autem 259733675932 in tabula Logarithmorum quæſita dabit figuras initiales numeri quæſiti, quæ erunt 181858. Quanquam ergo iſte numerus nullo modo exhiberi queat, tamen affirmari poteſt eum omnino ex 5050446 figuris conſtare, atque figuras initiales ſex eſſe 181858, quas dextrorſum adhuc 5050440 figuræ ſequantur, quarum inſuper nonnullæ ex majori Logarithmorum canone definiri poſſent, undecim ſcilicet figuræ initiales erunt 18185852986. CAP. VI.

CAPUT VII.

De quantitatum exponentialium ac Logarithmorum per Series explicatione.

114. Quia eſt $a^0 = 1$, atque creſcente Exponente ipſius a ſimul valor Poteſtatis augetur, ſi quidem a eſt numerus unitate major; ſequitur ſi Exponens infinite parum cyphram excedat, Poteſtatem ipſam quoque infinite parum unitatem eſſe ſuperaturam. Sit ω numerus infinite parvus, ſeu Fractio tam exigua, ut tantum non nihilo ſit æqualis, erit $a^\omega = 1 + \psi$, exiſtente ψ quoque numero infinite parvo. Ex præcedente enim capite conſtat niſi ψ eſſet numerus infinite parvus, neque ω talem eſſe poſſe. Erit ergo vel $\psi = \omega$, vel $\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ a pendebit, quæ cum adhuc ſit incognita, ponatur $\psi = k\omega$, ita ut ſit $a^\omega = 1 + k\omega$; &, ſumta a pro baſi Logarithmica, erit $\omega = l(1 + k\omega)$.

EXEMPLUM.

Quo clarius appareat, quemadmodum numerus k pendeat a baſi a, ponamus eſſe $a = 10$; atque ex tabulis vulgaribus quæramus Logarithmum numeri quam minime unitatem ſupe-

LIB. I. rantis, puta $1 + \frac{1}{1000000}$, ita ut sit $k\omega = \frac{1}{1000000}$; erit $l(1 + \frac{1}{1000000}) = l\frac{1000001}{1000000} = 0,0000043429 = \omega$. Hinc, ob $k\omega = 0,00000100000$, erit $\frac{1}{k} = \frac{43429}{100000}$ & $k = \frac{100000}{43429} = 2,30258$: unde patet k esse numerum finitum pendentem a valore basis a. Si enim alius numerus pro basi a statuatur, tum Logarithmus ejusdem numeri $1 + k\omega$ ad priorem datam tenebit rationem, unde simul alius valor litteræ k prodiret.

115. Cum sit $a^{\omega} = 1 + k\omega$, erit $a^{i\omega} = (1 + k\omega)^i$, quicunque numerus loco i substituatur. Erit ergo $a^{i\omega} = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{1.2}k^2\omega^2 + \frac{i(i-1)(i-2)}{1.2.3}k^3\omega^3 +$ &c. Quod si ergo statuatur $i = \frac{z}{\omega}$, & z denotet numerum quemcunque finitum, ob ω numerum infinite parvum, fiet i numerus infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut sit ω Fractio denominatorem habens infinitum, adeoque infinite parva, qualis est assumta. Substituatur ergo $\frac{z}{i}$ loco ω, eritque $a^z = (1 + \frac{kz}{i})^i = 1 + \frac{1}{1}kz + \frac{1(i-1)}{1.2i}k^2z^2 + \frac{1(i-1)(i-2)}{1.2i.3i}k^3z^3 + \frac{1(i-1)(i-2)(i-3)}{1.2i.3i.4i}k^4z^4 +$ &c., quæ æquatio erit vera si pro i numerus infinite magnus substituatur. Tum vero est k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i sit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; patet enim quo major numerus loco i substituatur, eo propius valorem Fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum, hinc si

i ſit numerus omni aſſignabili major, Fractio quoque $\frac{i-1}{i}$ CAP. VII.
ipſam unitatem adæquabit. Ob ſimilem autem rationem erit $\frac{i-2}{i}=1$; $\frac{i-3}{i}=1$; & ita porro; hinc ſequitur fore $\frac{i-1}{2i}=\frac{1}{2}$; $\frac{i-2}{3i}=\frac{1}{3}$; $\frac{i-3}{4i}=\frac{1}{4}$; & ita porro. His igitur valoribus ſubſtitutis, erit $a^z=1+\frac{kz}{1}+\frac{k^2z^2}{1.2}+\frac{k^3z^3}{1.2.3}+\frac{k^4z^4}{1.2.3.4}+$ &c. in infinitum. Hæc autem æquatio ſimul relationem inter numeros a & k oſtendit, poſito enim $z=1$, erit $a=1+\frac{k}{1}+\frac{k^2}{1.2}+\frac{k^3}{1.2.3}+\frac{k^4}{1.2.3.4}+$ &c., atque ut a ſit $=10$, neceſſe eſt ut ſit circiter $k=2,30258$, uti ante invenimus.

117. Ponamus eſſe $b=a^n$, erit, ſumto numero a pro baſi Logarithmica, $lb=n$. Hinc, cum ſit $b^z=a^{nz}$, erit per Seriem infinitam $b^z=1+\frac{knz}{1}+\frac{k^2n^2z^2}{1.2}+\frac{k^3n^3z^3}{1.2.3}+\frac{k^4n^4z^4}{1.2.3.4}+$ &c., poſito vero lb pro n, erit $b^z=1+\frac{kz}{1}lb+\frac{k^2z^2}{1.2}(lb)^2+\frac{k^3z^3}{1.2.3}(lb)^3+\frac{k^4z^4}{1.2.3.4}(lb)^4+$ &c.. Cognito ergo valore litteræ k ex dato valore baſis a, quantitas exponentialis quæcunque b^z per Seriem infinitam exprimi poterit, cujus termini ſecundum Poteſtates ipſius z procedant. His expoſitis oſtendamus quoque quomodo Logarithmi per Series infinitas explicari poſſint.

118. Cum ſit $a^\omega=1+k\omega$, exiſtente ω Fractione infinite parva, atque ratio inter a & k definiatur per hanc æquationem $a=1+\frac{k}{1}+\frac{k^2}{1.2}+\frac{k^3}{1.2.3}+$ &c., ſi a ſumatur pro baſi Logarithmica, erit $\omega=l(1+k\omega)$ & $i\omega=l(1+k\omega)^i$.

Manifestum autem est, quo major numerus pro i sumatur, eo magis Potestatem $(1+k\omega)^i$ unitatem esse superaturam; atque statuendo $i =$ numero infinito, valorem Potestatis $(1+k\omega)^i$ ad quemvis numerum unitate majorem ascendere. Quod si ergo ponatur $(1+k\omega)^i = 1+x$, erit $l(1+x) = i\omega$, unde, cum sit $i\omega$ numerus finitus, Logarithmus scilicet numeri $1+x$, perspicuum est, i esse debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non posset.

119. Cum autem positum sit $(1+k\omega)^i = 1+x$, erit $1+k\omega = (1+x)^{\frac{1}{i}}$ & $k\omega = (1+x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k}((1+x)^{\frac{1}{i}} - 1)$. Quia vero est $i\omega = l(1+x)$, erit $l(1+x) = \frac{i}{k}(1+x)^{\frac{1}{i}} - \frac{i}{k}$, posito i numero infinite magno. Est autem $(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i.\ 2i}x^2 + \frac{1(i-1)(2i-1)}{i.\ 2i.\ 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i.\ 2i.\ 3i.\ 4i}x^4 +$ &c. Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{2i-1}{3i} = \frac{2}{3}$; $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1+x)^{\frac{i}{1}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c., & consequenter $l(1+x) = \frac{1}{k}(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c.$)$, posita basi Logarithmica $= a$ denotante k numerum huic basi convenientem, ut scilicet sit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c.

120. Cum igitur habeamus Seriem Logarithmo numeri $1+x$ æqualem, ejus ope ex data basi a definire poterimus valorem numeri

numeri k. Si enim ponamus $1+x=a$, ob $la=1$, erit CAP. VII.
$1=\frac{1}{k}(\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.)$, hincque habebitur $k=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.$, cujus ideo Seriei infinitæ valor, si ponatur $a=10$, circiter esse debebit $=2,30258$; quanquam difficulter intelligi potest esse $2,30258=\frac{9}{1}-\frac{9^2}{2}+\frac{9^3}{3}-\frac{9^4}{4}+\&c.$, quoniam hujus Seriei termini continuo fiunt majores, neque adeo aliquot terminis sumendis summa vero propinqua haberi potest: cui incommodo mox remedium afferetur.

121. Quoniam igitur est $l(1+x)=\frac{1}{k}(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\&c.)$, erit, posito x negativo, $l(1-x)=-\frac{1}{k}(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\&c.)$. Subtrahatur Series posterior a priori, erit $l(1+x)-l(1-x)=l\frac{1+x}{1-x}=\frac{2}{k}\times(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\&c.)$. Nunc ponatur $\frac{1+x}{1-x}=a$, ut sit $x=\frac{a-1}{a+1}$, ob $la=1$ erit $k=2(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\&c.)$, ex qua æquatione valor numeri k ex basi a inveniri poterit. Si ergo basis a ponatur $=10$ erit $k=2(\frac{9}{11}+\frac{9^3}{3.11^3}+\frac{9^5}{5.11^5}+\frac{9^7}{7.11^7}+\&c.)$, cujus Seriei termini sensibiliter decrescunt, ideoque mox valorem pro k satis propinquum exhibent.

122. Quoniam ad systema Logarithmorum condendum basin a pro lubitu accipere licet, ea ita assumi poterit ut fiat $k=1$. Ponamus ergo esse $k=1$, eritque per Seriem supra

 (116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$, qui termini, si in fractiones decimales convertantur atque actu addantur, præbebunt hunc valorem pro $a = 2,71828182845904523536028$, cujus ultima adhuc nota veritati est consentanea. Quod si jam ex hac basi Logarithmi construantur, ii vocari solent Logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolæ per istiusmodi Logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc $2,718281828459$ &c. constanter litteram e, quæ ergo denotabit basin Logarithmorum naturalium seu hyperbolicorum, cui respondet valor litteræ $k = 1$; sive hæc littera e quoque exprimet summam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$ in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1 + \omega$ Logarithmus sit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innotescat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt. Erit ergo, posita e pro numero supra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$ ipsi vero Logarithmi hyperbolici ex his Seriebus invenientur, quibus est $l(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$, & $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \&c.$, quæ Series vehementer convergunt, si pro x statuatur fractio valde parva: ita ex Serie posteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum. Posito namque $x = \frac{1}{5}$, erit $l\frac{6}{4} = l\frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} + \&c.$, & facto $x = \frac{1}{7}$, erit $l\frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} +$

$\frac{2}{7\cdot 7^7} +$ &c., facto $x = \frac{1}{9}$, erit $l\frac{5}{4} = \frac{2}{1\cdot 9} + \frac{2}{3\cdot 9^3} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} +$ &c.. Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l\frac{3}{2} + l\frac{4}{3} = l2$; tum $l\frac{3}{2} + l2 = l3$; & $2l2 = l4$; porro $l\frac{5}{4} + l4 = l5$; $l2 + l3 = l6$; $3l2 = l8$; $2l3 = l9$; & $l2 + l5 = l10$. CAP. VII.

EXEMPLUM.

Hinc Logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

$l1$	=	0, 00000	00000	00000	00000	00000
$l2$	=	0, 69314	71805	59945	30941	72321
$l3$	=	1, 09861	22886	68109	69139	52452
$l4$	=	1, 38629	43611	19890	61883	44642
$l5$	=	1, 60943	79124	34100	37460	07593
$l6$	=	1, 79175	94692	28055	00081	24773
$l7$	=	1, 94591	01490	55313	30510	54639
$l8$	=	2, 07944	15416	79835	92825	16964
$l9$	=	2, 19722	45773	36219	38279	04905
$l10$	=	2, 30258	50929	94045	68401	79914

Hi scilicet Logarithmi omnes ex superioribus tribus Seriebus sunt deducti, præter $l7$, quem hoc compendio sum assecutus. Posui nimirum in Serie posteriori $x = \frac{1}{99}$ sicque obtinui $l\frac{100}{98} = l\frac{50}{49} = 0,02020270731751944840078230$, qui subtractus a $l50 = 2l5 + l2 = 3,9120230054281460586187508$, relinquit $l49$, cujus semissis dat $l7$.

M 2

LIB. I. 124. Ponatur Logarithmus hyperbolicus ipsius $1+x$ seu $l(1+x)=y$; erit $y=\frac{x}{1}-\frac{x}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$ &c.. Sumto autem numero a pro basi Logarithmica, sit numeri ejusdem $1+x$ Logarithmus $=v$; erit, ut vidimus, $v=\frac{1}{k}(x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.)=\frac{y}{k}$; hincque $k=\frac{y}{v}$; ex quo commodissime valor ipsius k basi a respondens ita definitur ut sit æqualis cujusvis numeri Logarithmo hyperbolico diviso per Logarithmum ejusdem numeri ex basi a formatum. Posito ergo numero hoc $=a$, erit $v=1$, hincque fit $k=$ Logarithmo hyperbolico basis a. In systemate ergo Logarithmorum communium, ubi est $a=10$, erit $k=$ Logarithmo hyperbolico ipsius 10, unde fit $k=2,30258509299404568401799 14$, quem valorem jam supra satis prope collegimus. Si ergo singuli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem $0,43429448190325182765 11289$, prodibunt Logarithmi vulgares basi $a=10$ convenientes.

125. Cum sit $e^z=1+\frac{z}{1}+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+$ &c., si ponatur $a^y=e^z$, erit, sumtis Logarithmis hyperbolicis, $y\,la=z$, quia est $le=1$, quo valore loco z substituto, erit $a^y=1+\frac{y\,la}{1}+\frac{y^2(la)^2}{1.2}+\frac{y^3(la)^3}{1.2.3}+$ &c., unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari potest. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per potestates exponi possunt. Erit enim $e^z=(1+\frac{z}{i})^i$, hincque $a^y=(1+\frac{y\,la}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1+x)=i((1+x)^{\frac{1}{i}}-1)$. De cetero

Logarithmorum hyperbolicorum ufus in calculo integrali fufius demonftrabitur.

CAPUT VIII.

De quantitatibus tranfcendentibus ex Circulo ortis.

126. Post Logarithmos & quantitates exponentiales confiderari debent Arcus circulares eorumque Sinus & Cofinus, quia non folum aliud quantitatum tranfcendentium genus conftituunt, fed etiam ex ipfis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli feu Sinum totum effe $= 1$, atque fatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poffe, per approximationes autem inventa eft Semicircumferentia hujus Circuli effe $=$ 3, 14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384446 +, pro quo numero, brevitatis ergo, fcribam π, ita ut fit $\pi =$ Semicircumferentiæ Circuli, cujus Radius $= 1$, feu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo affumo $= 1$; hujus Arcus z confiderari potiffimum folent Sinus & Cofinus. Sinum autem Arcus z in pofterum hoc modo indicabo *fin. A.* z, feu tantum *fin.* z. Cofinum vero hoc modo *cof. A.* z, feu tantum *cof.* z. Ita, cum π fit Arcus 180°, erit $fin.\, 0\,\pi = 0$, $cof.\, 0\,\pi = 1$; & $fin.\, \frac{1}{2}\pi = 1$, $cof.\, \frac{1}{2}\pi = 0$; $fin.\, \pi = 0$, $cof.\, \pi = -1$; $fin.\, \frac{3}{2}\pi = -1$, $cof.\, \frac{3}{2}\pi = 0$; $fin.\, 2\pi = 0$ & $cof.\, 2\pi = 1$. Omnes ergo Sinus & Cofinus intra limites $+1$ & -1 con-

LIB. I. tinentur. Erit autem porro $cof.\, z = fin.\, (\frac{1}{2}\pi - z)$, & $fin.\, z = cof.\, (\frac{1}{2}\pi - z)$, atque $(fin.\, z)^2 + (cof.\, z)^2 = 1$. Præter has denominationes notandæ ſunt quoque hæ: $tang.\, z$, quæ denotat Tangentem Arcus z; $cot.\, z$ Cotangentem Arcus z; conſtatque eſſe $tang.\, z = \frac{fin.\, z}{cof.\, z}$ & $cot.\, z = \frac{cof.\, z}{fin.\, z} = \frac{1}{tang.\, z}$; quæ omnia ex Trigonometria ſunt nota.

128. Hinc vero etiam conſtat ſi habeantur duo Arcus y & z, fore $fin.\, (y + z) = fin.\, y.\, cof.\, z + cof.\, y.\, fin.\, z$ & $cof.\, (y + z) = cof.\, y.\, cof.\, z - fin.\, y.\, fin.\, z$, itemque $fin.\, (y - z) = fin.\, y.\, cof.\, z - cof.\, y.\, fin.\, z$ & $cof.\, (y - z) = cof.\, y.\, cof.\, z + fin.\, y.\, fin.\, z$.

Hinc loco y ſubſtituendo Arcus $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, &c., erit

$fin.\, (\frac{1}{2}\pi + z) = + cof.\, z$	$fin.\, (\frac{1}{2}\pi - z) = + cof.\, z$
$cof.\, (\frac{1}{2}\pi + z) = - fin.\, z$	$cof.\, (\frac{1}{2}\pi - z) = + fin.\, z$
$fin.\, (\pi + z) = - fin.\, z$	$fin.\, (\pi - z) = + fin.\, z$
$cof.\, (\pi + z) = - cof.\, z$	$cof.\, (\pi - z) = - cof.\, z$
$fin.\, (\frac{3}{2}\pi + z) = - cof.\, z$	$fin.\, (\frac{3}{2}\pi - z) = - cof.\, z$
$cof.\, (\frac{3}{2}\pi + z) = + fin.\, z$	$cof.\, (\frac{3}{2}\pi - z) = - fin.\, z$
$fin.\, (2\pi + z) = + fin.\, z$	$fin.\, (2\pi - z) = - fin.\, z$
$cof.\, (2\pi + z) = + cof.\, z$	$cof.\, (2\pi - z) = + cof.\, z$

Si ergo n denotet numerum integrum quemcunque, erit CAP. VIII.

$\sin.\left(\frac{4n+1}{2}\pi+z\right)=+\cos.z$	$\sin.\left(\frac{4n+1}{2}\pi-z\right)=+\cos.z$
$\cos.\left(\frac{4n+1}{2}\pi+z\right)=-\sin.z$	$\cos.\left(\frac{4n+1}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+2}{2}\pi+z\right)=-\sin.z$	$\sin.\left(\frac{4n+2}{2}\pi-z\right)=+\sin.z$
$\cos.\left(\frac{4n+2}{2}\pi+z\right)=-\cos.z$	$\cos.\left(\frac{4n+2}{2}\pi-z\right)=-\cos.z$
$\sin.\left(\frac{4n+3}{2}\pi+z\right)=-\cos.z$	$\sin.\left(\frac{4n+3}{2}\pi-z\right)=-\cos.z$
$\cos.\left(\frac{4n+3}{2}\pi+z\right)=+\sin.z$	$\cos.\left(\frac{4n+3}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+4}{2}\pi+z\right)=+\sin.z$	$\sin.\left(\frac{4n+4}{2}\pi-z\right)=-\sin.z$
$\cos.\left(\frac{4n+4}{2}\pi+z\right)=+\cos.z$	$\cos.\left(\frac{4n+4}{2}\pi-z\right)=+\cos.z$

Quæ formulæ veræ sunt sive n sit numerus affirmativus sive negativus integer.

129. Sit $\sin. z = p$ & $\cos. z = q$, erit $pp + qq = 1$; & $\sin. y = m$ & $\cos. y = n$, ut sit quoque $mm + nn = 1$; Arcuum ex his compositorum Sinus & Cosinus ita se habebunt.

$\sin. z = p$	$\cos. z = q$
$\sin.(y+z) = mq + np$	$\cos.(y+z) = nq - mp$
$\sin.(2y+z) = 2mnq + (nn - mm)p$	$\cos.(2y+z) = (nn - mm)q - 2mnp$
$\sin.(3y+z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$	$\cos.(3y+z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$
&c.	&c.

Arcus isti z, $y+z$, $2y+z$, $3y+z$, &c., in arithmetica progressione progrediuntur; eorum vero tam Sinus quam Cosinus progressionem recurrentem constituunt, qualis ex denominatore $1 - 2nx + (mm+nn)xx$ oritur; est enim

LIB. I. $fin.(2y+z)=2n.fin.(y+z)-(mm+nn)fin.z$ five
$fin.(2y+z)=2cof.y.fin.(y+z)-(fin.z)$; atque fimili modo
$cof.(2y+z)=2cof.y.cof.(y+z)-cof.z$. Eodem modo erit porro
$fin.(3y+z)=2cof.y.fin.(2y+z)-fin.(y+z)$, &
$cof.(3y+z)=2cof.y.cof.(2y+z)-cof.(y+z)$, itemque
$fin.(4y+z)=2cof.y.fin.(3y+z)-fin.(2y+z)$, &
$cof.(4y+z)=2cof.y.cof.(3y+z)-cof.(2y+z)$, &c.
Cujus legis beneficio Arcuum in progreffione arithmetica progredientium tam Sinus quam Cofinus quoufque libuerit expedite formari poffunt.

130. Cum fit $fin.(y+z)=fin.y.cof.z+cof.y.fin.z$, atque $fin.(y-z)=fin.y.cof.z-cof.y.fin.z$, erit his expreffionibus vel addendis vel fubtrahendis:

$$fin.y.cof.z=\frac{fin.(y+z)+fin.(y-z)}{2}$$

$$cof.y.fin.z=\frac{fin.(y+z)-fin.(y-z)}{2}$$

Quia porro eft $cof.(y+z)=cof.y.cof.z-fin.y.fin.z$, atque $cof.(y-z)=cof.y.cof.z+fin.y.fin.z$, erit pari modo

$$cof.y.cof.z.=\frac{cof.(y-z)+cof.(y+z)}{2}$$

$$fin.y.fin.z.=\frac{cof.(y-z)-cof.(y+z)}{2}.$$

Sit $y=z=\frac{1}{2}v$, erit ex his poftremis formulis:

$$(cof.\tfrac{1}{2}v)^2=\frac{1+cof.v}{2},\ \&\ cof.\tfrac{1}{2}v=\sqrt{\frac{1+cof.v}{2}}$$

$$(fin.\tfrac{1}{2}v)^2=\frac{1-cof.v}{2},\ \&\ fin.\tfrac{1}{2}v=\sqrt{\frac{1-cof.v}{2}}$$

unde, ex dato Cofinu cujufque anguli reperiuntur ejus femiffis Sinus & Cofinus.

131. Ponatur Arcus $y+z=a$, & $y-z=b$, erit $y=\frac{a+b}{2}$ & $z=\frac{a-b}{2}$; quibus in fuperioribus formulis fubfti-

tutis

tutis, habebuntur hæ æquationes, tanquam totidem Theoremata. CAP. VIII.

$$fin.\,a + fin.\,b = 2\,fin.\,\frac{a+b}{2}\,cof.\,\frac{a-b}{2}$$
$$fin.\,a - fin.\,b = 2\,cof.\,\frac{a+b}{2}\,fin.\,\frac{a-b}{2}$$
$$cof.\,a + cof.\,b = 2\,cof.\,\frac{a+b}{2}\,cof.\,\frac{a-b}{2}$$
$$cof.\,b - cof.\,a = 2\,fin.\,\frac{a+b}{2}\,fin.\,\frac{a-b}{2}.$$

Ex his porro nafcuntur, ope divifionis, hæc Theoremata

$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} = tang.\,\frac{a+b}{2}\,cot.\,\frac{a-b}{2} = \frac{tang.\,\frac{a+b}{2}}{tang.\,\frac{a-b}{2}}$$
$$\frac{fin.\,a + fin.\,b}{cof.\,a + cof.\,b} = tang.\,\frac{a+b}{2}$$
$$\frac{fin.\,a + fin.\,b}{cof.\,b - cof.\,a} = cot.\,\frac{a-b}{2}$$
$$\frac{fin.\,a - fin.\,b}{cof.\,a + cof.\,b} = tang.\,\frac{a-b}{2}$$
$$\frac{fin.\,a - fin.\,b}{cof.\,b - cof.\,a} = cot.\,\frac{a+b}{2}$$
$$\frac{cof.\,a + cof.\,b}{cof.\,b - cof.\,a} = cot.\,\frac{a+b}{2}.\,cot.\,\frac{a-b}{2}$$

Ex his denique deducuntur ifta Theoremata

$$\frac{fin.\,a + fin.\,b}{cof.\,a + cof.\,b} = \frac{cof.\,b - cof.\,a}{fin.\,a - fin.\,b};$$
$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} \times \frac{cof.\,a + cof.\,b}{cof.\,b - cof.\,a} = \left(cot.\,\frac{a-b}{2}\right)^2$$
$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} \times \frac{cof.\,b - cof.\,a}{cof.\,a + cof.\,b} = \left(tang.\,\frac{a+b}{2}\right)^2.$$

132. Cum fit $(fin.\,z)^2 + (cof.\,z)^2 = 1$ erit, Factoribus fumptis, $(cof.\,z + \sqrt{-1}.\,fin.\,z)\,(cof.\,z - \sqrt{-1}.\,fin.\,z) = 1$; qui Factores, etfi imaginarii, tamen ingentem præftant ufum in Arcubus combinandis & multiplicandis. Quæratur enim productum horum Factorum $(cof.\,z + \sqrt{-1}.\,fin.\,z)(cof.\,y + \sqrt{-1}.\,fin.\,y)$ ac reperietur $cof.\,y.\,cof.\,z - fin.\,y.\,fin.\,z + (cof.\,y.\,fin.\,z + fin.\,y.\,cof.\,z)$

LIB. I. $\sqrt{-1}$. Cum autem sit $cos. y. cos. z - sin. y. sin. z = cos. (y+z)$ & $cos. y. sin. z + sin. y. cos. z = sin. (y+z)$ erit hoc productum $(cos. y + \sqrt{-1}. sin. y)(cos. z + \sqrt{-1}. sin. z) = cos. (y+z) + \sqrt{-1}. sin. (y+z)$

& simili modo

$(cos. y - \sqrt{-1}. sin. y)(cos. z - \sqrt{-1}. sin. z) = cos. (y+z) - \sqrt{-1}. sin. (y+z)$

item

$(cos. x \pm \sqrt{-1}. sin. x)(cos. y \pm \sqrt{-1}. sin. y)(cos. z \pm \sqrt{-1}. sin. z) = cos. (x+y+z) \pm \sqrt{-1}. sin. (x+y+z)$.

133. Hinc itaque sequitur fore $(cos. z \pm \sqrt{-1}. sin. z)^2 = cos. 2z \pm \sqrt{-1}. sin. 2z$, & $(cos. z \pm \sqrt{-1}. sin. z)^3 = cos. 3z \pm \sqrt{-1}. sin. 3z$;

ideoque generaliter erit $(cos. z \pm \sqrt{-1}. sin. z)^n = cos. nz \pm \sqrt{-1}. sin. nz$:

Unde, ob signorum ambiguitatem, erit

$$cos. nz = \frac{(cos. z + \sqrt{-1}. sin. z)^n + (cos. z - \sqrt{-1}. sin. z)^n}{2}$$ &

$$sin. nz = \frac{(cos. z + \sqrt{-1}. sin. z)^n - (cos. z - \sqrt{-1}. sin. z)^n}{2\sqrt{-1}}$$

Evolutis ergo binomiis hisce erit per Series:

$$cos. nz = (cos. z)^n - \frac{n(n-1)}{1.2}(cos. z)^{n-2}(sin. z)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(cos. z)^{n-4}(sin. z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}(cos. z)^{n-6}(sin. z)^6 + \&c.,$$ &

$$sin. nz = \frac{n}{1}(cos. z)^{n-1} sin. z - \frac{n(n-1)(n-2)}{1.2.3}(cos. z)^{n-3}(sin. z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(cos. z)^{n-5}(sin. z)^5 - \&c.$$

134. Sit Arcus z infinite parvus, erit $\mathit{fin.}\ z = z$ & $\mathit{cof.}\ z = 1$: fit autem n numerus infinite magnus, ut fit Arcus nz finitæ magnitudinis, puta, $nz = v$; ob $\mathit{fin.}\ z = z = \frac{v}{n}$ erit CAP. VIII.

$$\mathit{cof.}\ v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c.,\ \&$$

$$\mathit{fin.}\ v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c.$$

Dato ergo Arcu v, ope harum Serierum ejus Sinus & Cofinus inveniri poterunt; quarum formularum ufus quo magis pateat, ponamus Arcum v effe ad quadrantem, feu ad arcum 90°, ut m ad n, feu effe $v = \frac{m}{n} \cdot \frac{\pi}{2}$; Quia nunc valor ipfius π conftat, fi is ubique fubftituatur, prodibit

$$\mathit{fin.}\ A.\ \frac{m}{n}\ 90^\circ =$$

$$\begin{aligned}
&+ \tfrac{m}{n} \cdot 1{,}5707963267948966192313216916 \\
&- \tfrac{m^3}{n^3} \cdot 0{,}6459640975062462536557565636 \\
&+ \tfrac{m^5}{n^5} \cdot 0{,}0796926262461670451205055488 \\
&- \tfrac{m^7}{n^7} \cdot 0{,}0046817541353186881006854632 \\
&+ \tfrac{m^9}{n^9} \cdot 0{,}0001604411847873598218726605 \\
&- \tfrac{m^{11}}{n^{11}} \cdot 0{,}0000035988432352120853404580 \\
&+ \tfrac{m^{13}}{n^{13}} \cdot 0{,}0000000569217292196792681171 \\
&- \tfrac{m^{15}}{n^{15}} \cdot 0{,}0000000006688035109811467224 \\
&+ \tfrac{m^{17}}{n^{17}} \cdot 0{,}0000000000060669357311061950
\end{aligned}$$

$$
\begin{aligned}
&- \frac{m^{19}}{n^{19}} \cdot 0,0000000000000437706546731370 \\
&+ \frac{m^{21}}{n^{21}} \cdot 0,0000000000000002571422892856 \\
&- \frac{m^{23}}{n^{23}} \cdot 0,0000000000000000012538995403 \\
&+ \frac{m^{25}}{n^{25}} \cdot 0,0000000000000000000051564550 \\
&- \frac{m^{27}}{n^{27}} \cdot 0,0000000000000000000000181239 \\
&+ \frac{m^{29}}{n^{29}} \cdot 0,0000000000000000000000000549,
\end{aligned}
$$

atque $cos.\ A. \frac{m}{n} 90^\circ =$

$$
\begin{aligned}
&+ \frac{m^{0}}{n^{0}} \cdot 1,0000000000000000000000000000 \\
&- \frac{m^{2}}{n^{2}} \cdot 1,2337005501361698273543113745 \\
&+ \frac{m^{4}}{n^{4}} \cdot 0,2536695079010480136365633659 \\
&- \frac{m^{6}}{n^{6}} \cdot 0,0208634807633529608730516364 \\
&+ \frac{m^{8}}{n^{8}} \cdot 0,0009192602748394265802417158 \\
&- \frac{m^{10}}{n^{10}} \cdot 0,0000252020423730606054810526 \\
&+ \frac{m^{12}}{n^{12}} \cdot 0,0000004710874778818171503665 \\
&- \frac{m^{14}}{n^{14}} \cdot 0,0000000063866030837918522408 \\
&+ \frac{m^{16}}{n^{16}} \cdot 0,0000000000656596311497947230 \\
&- \frac{m^{18}}{n^{18}} \cdot 0,0000000000005294400200734620 \\
&+ \frac{m^{20}}{n^{20}} \cdot 0,0000000000000034377391790981 \\
&- \frac{m^{22}}{n^{22}} \cdot 0,0000000000000000183599165212
\end{aligned}
$$

$+ \frac{m^{24}}{n^{24}}.\ 0,0000000000000000000820675327$ CAP. VIII.

$- \frac{m^{26}}{n^{26}}.\ 0,0000000000000000000003115285$

$+ \frac{m^{28}}{n^{28}}.\ 0,0000000000000000000000010165$

$- \frac{m^{30}}{n^{30}}.\ 0,0000000000000000000000000026.$

Cum igitur ſufficiat Sinus & Coſinus angulorum uſque ad 45° noſſe, fractio $\frac{m}{n}$ ſemper minor erit quam $\frac{1}{2}$, hincque etiam ob Poteſtates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini ſufficiant, præcipue, ſi Sinus & Coſinus non ad tot figuras perducti deſiderentur.

135. Inventis Sinibus & Coſinibus inveniri quidem poſſunt Tangentes & Cotangentes, per analogias conſuetas, at quia in hujuſmodi ingentibus numeris multiplicatio & diviſio vehementer eſt moleſta, peculiari modo eas exprimere convenit. Erit ergo

$$tang.\, v = \frac{\text{ſin.}\, v}{\text{coſ.}\, v} = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3\ldots6} + \&c.}$$

$$\&\ cot.\, v = \frac{\text{coſ.}\, v}{\text{ſin.}\, v} = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3\ldots6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots7} + \&c.}$$

si jam sit Arcus $v = \frac{m}{n}\, 90^\circ$ erit eodem modo quo ante

tang. A. $\frac{m}{n}\, 90^\circ =$		*cot.* A. $\frac{m}{n}\, 90^\circ =$	
$+\frac{2mn}{nn-mm}$.	0, 6366197723675	$+\frac{n}{m}$.	0, 6366197723675
$+\frac{m}{n}$.	0, 2975567820597	$-\frac{4mn}{4nn-mm}$.	0, 3183098861837
$+\frac{m^3}{n^3}$.	0, 0186886502773	$-\frac{m}{n}$.	0, 2052888894145
$+\frac{m^5}{n^5}$.	0, 0018424752034	$-\frac{m^3}{n^3}$.	0, 0065510747882
$+\frac{m^7}{n^7}$.	0, 0001975800714	$-\frac{m^5}{n^5}$.	0, 0003450292554
$+\frac{m^9}{n^9}$.	0, 0000216977245	$-\frac{m^7}{n^7}$.	0, 0000202791060
$+\frac{m^{11}}{n^{11}}$.	0, 0000024011370	$-\frac{m^9}{n^9}$.	0, 0000012366527
$+\frac{m^{13}}{n^{13}}$.	0, 0000002664132	$-\frac{m^{11}}{n^{11}}$.	0, 0000000764959
$+\frac{m^{15}}{n^{15}}$.	0, 0000000295864	$-\frac{m^{13}}{n^{13}}$.	0, 0000000047597
$+\frac{m^{17}}{n^{17}}$.	0, 0000000032867	$-\frac{m^{15}}{n^{15}}$.	0, 0000000002969
$+\frac{m^{19}}{n^{19}}$.	0, 0000000003651	$-\frac{m^{17}}{n^{17}}$.	0, 0000000000185
$+\frac{m^{21}}{n^{21}}$.	0, 0000000000405	$-\frac{m^{19}}{n^{19}}$.	0, 0000000000011
$+\frac{m^{23}}{n^{23}}$.	0, 0000000000045		
$+\frac{m^{25}}{n^{25}}$.	0, 0000000000005		

quarum Serierum ratio infra fusius exponetur.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum Sinus & Cosinus, inde simul omnium angulorum majorum Sinus & Cosinus haberi. Verum si tantum angulorum 30° minorum habeantur

Sinus & Cosinus, ex iis, per solam additionem & subtractionem, omnium angulorum majorum Sinus & Cosinus inveniri possunt. Cum enim sit $fin.\ 30^\circ = \frac{1}{2}$, erit, posito $y = 30^\circ$ (ex §. 130) $cof.\ z = fin.\ (30 + z) + fin.\ (30 - z)$; & $fin.\ z = cof.\ (30 - z) - cof.\ (30 + z)$, ideoque ex Sinibus & Cosinibus angulorum z & $30 - z$, reperiuntur $fin.\ (30 + z) = cof.\ z - fin.\ (30 - z)$ & $cof.\ (30 + z) = cof.\ (30 - z) - fin.\ z$, unde Sinus & Cosinus angulorum a 30° ad 60°, hincque omnes majores definiuntur. CAP. VIII.

137. In Tangentibus & Cotangentibus simile subsidium usu venit. Cum enim sit $tang.\ (a + b) = \frac{tang.\ a + tang.\ b}{1 - tang.\ a.\ tang.\ b}$, erit $tang.\ 2a = \frac{2\ tang.\ a}{1 - (tang.\ a)^2}$, & $cot.\ 2a = \frac{cot.\ a - tang.\ a}{2}$ unde ex Tangentibus & Cotangentibus Arcuum 30° minorum inveniuntur Cotangentes usque ad 60°.

Sit jam $a = 30 - b$ erit $2a = 60 - 2b$ & $cot.\ 2a = tang.\ (30 + 2b)$; erit ergo $tang.\ (30 + 2b) = \frac{cot.\ (30 - b) - tang.\ (30 - b)}{2}$, unde etiam Tangentes Arcuum 30° majorum obtinentur.

Secantes autem & Cosecantes ex Tangentibus per solam subtractionem inveniuntur; est enim $cofec.\ z = cot.\ \frac{1}{2} z - cot.\ z$, & hinc $fec.\ z = cot.\ (45^\circ - \frac{1}{2} z) - tang.\ z$. Ex his ergo luculenter perspicitur, quomodo canones Sinuum construi potuerint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $fin.\ z = \frac{v}{i}$ & $cof.\ z = 1$; his substitutis fit $cof.\ v =$

$$\frac{(1+\frac{v\sqrt{-1}}{i})^i+(1-\frac{v\sqrt{-1}}{i})^i}{2}$$; atque $fin.\ v =$

$$\frac{(1+\frac{v\sqrt{-1}}{i})^i-(1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}.$$ In Capite autem præcedente vidimus esse $(1+\frac{z}{i})^i = e^z$, denotante e basin Logarithmorum hyperbolicorum : scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $cof.\ v =$
$$\frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2}\ \&\ fin.\ v = \frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$
Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = cof.\ v + \sqrt{-1}.\ fin.\ v$ & $e^{-v\sqrt{-1}} = cof.\ v - \sqrt{-1}.\ fin.\ v$.

139. Sit jam in iisdem formulis §. 133, n numerus infinite parvus, seu $n = \frac{1}{i}$, existente i numero infinite magno, erit $cof.\ nz = cof.\ \frac{z}{i} = 1$ & $fin.\ nz = fin.\ \frac{z}{i} = \frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(cof.\ z+\sqrt{-1}.\ fin.\ z)^{\frac{1}{i}} + (cof.\ z-\sqrt{-1}.\ fin.\ z)^{\frac{1}{i}}}{2}\ \&$$

$$\frac{z}{i} = \frac{(cof.\ z+\sqrt{-1}.\ fin.\ z)^{\frac{1}{i}} - (cof.\ z-\sqrt{-1}.\ fin.\ z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$ Sumendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, seu $y^{\frac{1}{i}} = 1 + \frac{1}{i}ly$,

posito

posito y loco $1 + x$. Nunc igitur, posito loco y, partim $cos.z + \sqrt{-1}.\,sin.\,z$ partim $cos.\,z - \sqrt{-1}.\,sin.\,z$, prodibit $1 =$ CAP. VIII.

$$\frac{1 + \frac{1}{i}\,l(cos.z + \sqrt{-1}.sin.z) + 1 + \frac{1}{i}\,l(cos.z - \sqrt{-1}.sin.z)}{2}$$

$= 1$, ob Logarithmos evanescentes, ita ut hinc nil sequatur. Altera vero æquatio pro Sinu suppeditat:

$$\frac{z}{i} = \frac{\frac{1}{i}\,l(cos.z + \sqrt{-1}.sin.z) - \frac{1}{i}\,l(cos.z - \sqrt{-1}.sin.z)}{2\sqrt{-1}}$$

ideoque $z = \frac{1}{2\sqrt{-1}}\,l\frac{cos.z + \sqrt{-1}.\,sin.\,z}{cos.z - \sqrt{-1}.\,sin.\,z}$, unde patet quemadmodum Logarithmi imaginarii ad Arcus circulares revocentur.

140. Cum sit $\frac{sin.z}{cos.z} = tang.z$, Arcus z per suam Tangentem ita exprimetur ut sit $z = \frac{1}{2\sqrt{-1}}\,l\frac{1 + \sqrt{-1}.\,tang.z}{1 - \sqrt{-1}.\,tang.z}$. Supra vero (§. 123) vidimus esse $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} +$ &c.. Posito ergo $x = \sqrt{-1}.\,tang.z$, fiet $z = \frac{tang.z}{1} - \frac{(tang.z)^3}{3} + \frac{(tang.z)^5}{5} - \frac{(tang.z)^7}{7} +$ &c.. Si ergo ponamus $tang.z = t$, ut sit z Arcus, cujus Tangens est t, quem ita indicabimus $A.\,tang.\,t$, ideoque erit $z = A.\,tang.\,t$. Cognita ergo Tangente t erit Arcus respondens $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} -$ &c.. Cum igitur, si Tangens t æquetur Radio 1, fiat Arcus $z =$ Arcui 45°, seu $z = \frac{\pi}{4}$, erit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., quæ est Series a LEIBNITZIO primum producta, ad valorem Peripheriæ Circuli exprimendum.

141. Quo autem ex hujusmodi Serie longitudo Arcus Cir-

 culi expedite definiri possit, perspicuum est pro Tangente t fractionem satis parvam substitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus z, cujus Tangens t æquetur $\frac{1}{10}$, foret enim iste Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \&c.$, cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens est $= \frac{1}{10}$, ad totam Peripheriam tenet, non sit assignabilis. Hanc ob rem ad Peripheriam indagandam, ejusmodi Arcus quæri debet, qui sit simul pars aliquota Peripheriæ, & cujus Tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet Arcus $30°$ cujus Tangens est $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commensurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30° = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} - \&c.$, & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \&c.$, cujus Seriei ope valor ipsius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major est, quod primum singuli termini sint irrationales, tum vero quisque tantum, circiter, triplo sit minor quam præcedens. Huic itaque incommodo ita occurri poterit: sumatur Arcus $45°$ seu $\frac{\pi}{4}$ cujus valor, etsi per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$, exprimitur, tamen is retineatur, atque in duos Arcus a & b dispertiatur ut sit $a + b = \frac{\pi}{4} = 45°$. Cum igitur sit *tang.* $(a+b) = 1 = \frac{tang. a + tang. b}{1 - tang. a. \, tang. b}$ erit $1 - tang. a. \, tang. b =$

$tang. a + tang. b$ & $tang. b = \frac{1 - tang. a}{1 + tang. a}$. Sit nunc $tang. a = \frac{1}{2}$, erit $tang. b = \frac{1}{3}$, hinc uterque Arcus a & b per Seriem rationalem multo magis, quam superior, convergentem exprimetur, eorumqne summa dabit valorem Arcus $\frac{\pi}{4}$; hinc itaque erit CAP. VIII.

$$\pi = 4 \cdot \left\{ \begin{array}{l} \frac{1}{1.2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \frac{1}{9.2^9} - \&c. \\ \frac{1}{1.3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo semicircumferentiæ π inveniri potuisset, quam quidem factum est ope Seriei ante commemoratæ.

CAPUT IX.

De investigatione Factorum trinomialium.

143. QUEMADMODUM Factores simplices cujusque Functionis integræ inveniri oporteat, supra quidem ostendimus hoc fieri per resolutionem æquationum. Si enim proposita sit Functio quæcunque integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, hujusque quærantur Factores simplices formæ $p - qz$, manifestum est, si $p - qz$ fuerit Factor Functionis $\alpha + \beta z + \gamma z^2 + \&c.$, tum, posito $z = \frac{p}{q}$, quo casu Factor $p - qz$ fit $= 0$, etiam ipsam Functionem propositam evanescere debere. Hinc si $p - qz$ est Factor vel divisor Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, sequitur fore hanc

 expressionem $\alpha + \frac{\beta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\epsilon p^4}{q^4} + \&c. = 0$. Unde vicissim, si omnes radices $\frac{p}{q}$ hujus æquationis eruantur, singulæ dabunt totidem Factores simplices Functionis integræ propositæ $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, nempe $p - qz$. Patet autem simul numerum Factorum hujusmodi simplicium ex maxima Potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruuntur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope sæpenumero Factores simplices imaginarii inveniri queant. Quoniam vero Factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos Factores imaginarios reperiemus, si Factores investigemus duplices, seu hujus formæ $p - qz + rzz$, reales quidem, sed quorum Factores simplices sint imaginarii. Quod si enim Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, constent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, simul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores simplices habebit imaginarios, si fuerit $4pr > qq$ seu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Cosinus Angulorum sint unitate minores; formula $p - qz + rzz$ Factores simplices habebit imaginarios si fuerit $\frac{q}{2\sqrt{pr}} =$ Sinui vel Cosinui cujuspiam Anguli. Sit ergo $\frac{q}{2\sqrt{pr}} = cof. A\varphi$, seu $q = 2\sqrt{pr}.\ cof.\varphi$, atque trinomium $p - qz + rzz$ continebit Factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam $pp - 2pqz.\ cof.\varphi + qqzz$, cujus Factores simplices imaginarii erunt hi, $qz - p\,(cof.\varphi + \sqrt{-1}.\ fin.\varphi)$ & $qz - p\,(cof.\varphi - \sqrt{-1}.\ fin.\varphi)$. Ubi quidem patet si fuerit $cof.\varphi = \pm 1$, tum ambos Factores, ob $fin.\varphi = 0$, fieri æquales & reales.

146. Proposita ergo Functione integra $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ &c., ejus Factores simplices imaginarii eruentur, si determinentur litteræ p & q cum Angulo φ, ut hoc trinomium $pp - 2pqz.\ cof.\ \varphi + qqzz$ fiat Factor Functionis. Tum enim simul inerunt isti Factores simplices imaginarii $qz - p\,(cof.\varphi + \sqrt{-1}.\,fin.\,\varphi)$ & $qz - p\,(cof.\,\varphi - \sqrt{-1}.\,fin.\,\varphi)$. Quam ob rem Functio proposita evanescet, si ponatur tam $z = \frac{p}{q} \times (cof.\varphi + \sqrt{-1}.\,fin.\varphi)$ quam $z = \frac{p}{q}\,(cof.\,\varphi - \sqrt{-1}.\,fin.\,\varphi)$. Hinc, facta substitutione utraque, duplex nascetur æquatio, ex quibus tam fractio $\frac{p}{q}$ quam Arcus φ definiri poterunt.

147. Hæ autem substitutiones loco z faciendæ, etiamsi primo intuitu difficiles videantur, tamen per ea, quæ in Capite præcedente sunt tradita, satis expedite absolventur. Cum enim fuerit ostensum esse $(cof.\varphi \pm \sqrt{-1}.\,fin.\varphi)^n = cof.\,n\varphi \pm \sqrt{-1} \times fin.\,n\varphi$, sequentes formulæ loco singularum ipsius z Potestatum habebuntur substituendæ:

pro priori Factore	pro altero Factore
$z = \frac{p}{q}(cof.\ \varphi + \sqrt{-1}.fin.\ \varphi)$	$z = \frac{p}{q}(cof.\ \varphi - \sqrt{-1}.fin.\ \varphi)$
$z^2 = \frac{p^2}{q^2}(cof.2\varphi + \sqrt{-1}.fin.2\varphi)$	$z^2 = \frac{p^2}{q^2}(cof.2\varphi - \sqrt{-1}.fin.2\varphi)$
$z^3 = \frac{p^3}{q^3}(cof.3\varphi + \sqrt{-1}.fin.3\varphi)$	$z^3 = \frac{p^3}{q^3}(cof.3\varphi - \sqrt{-1}.fin.3\varphi)$
$z^4 = \frac{p^4}{q^4}(cof.4\varphi + \sqrt{-1}.fin.4\varphi)$	$z^4 = \frac{p^4}{q^4}(cof.4\varphi - \sqrt{-1}.fin.4\varphi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque substitutione sequentes duæ nascentur æquationes:

$$0 = \begin{cases} \alpha + \beta r.\ cof.\ \varphi + \gamma r^2.\ cof.\ 2\varphi + \delta r^3.\ cof.\ 3\varphi + \&c. \\ + \beta r\sqrt{-1}.\,fin.\varphi + \gamma r^2\sqrt{-1}.\,fin.2\varphi + \delta r^3\sqrt{-1}.\,fin.3\varphi + \&c. \end{cases}$$

$$0 = \begin{cases} \alpha + \beta r.\ cof.\ \varphi + \gamma r^2.\ cof.\ 2\varphi + \delta r^3.\ cof.\ 3\varphi + \&c. \\ - \beta r\sqrt{-1}.\,fin.\varphi - \gamma r^2\sqrt{-1}.\,fin.2\varphi - \delta r^3\sqrt{-1}.\,fin.3\varphi - \&c. \end{cases}$$

LIB. I. 148. Quod si hæ duæ æquationes invicem addantur & subtrahantur, & posteriori casu per $2\sqrt{-1}$ dividantur, prodibunt hæ duæ æquationes reales:

$$0 = \alpha + \beta r. \cos. \varphi + \gamma r^2. \cos. 2\varphi + \delta r^3. \cos. 3\varphi + \&c.$$
$$0 = \beta r. \sin. \varphi + \gamma r^2. \sin. 2\varphi + \delta r^3. \sin. 3\varphi + \&c.$$

quæ statim ex forma Functionis propositæ

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$$

formari possunt, ponendo primum pro unaquaque ipsius z potestate $z^n = r^n. \cos. n\varphi$, deinceps $z^n = r^n. \sin. n\varphi$. Sic enim ob $\sin. 0\varphi = 0$ & $\cos. 0\varphi = 1$, pro z^0 seu 1 in termino constanti priori casu ponitur 1, posteriori autem 0. Si ergo ex his duabus æquationibus definiantur incognitæ r & φ, ob $r = \frac{p}{q}$, habebitur Factor Functionis trinomialis $pp - 2pqz. \cos. \varphi + qqzz$, duos Factores simplices imaginarios involvens.

149. Si æquatio prior multiplicetur per $\sin. m\varphi$, posterior per $\cos. m\varphi$, atque producta vel addantur vel subtrahantur, prodibunt illæ duæ æquationes:

$$0 = \alpha. \sin. m\varphi + \beta r. \sin. (m+1)\varphi + \gamma r^2. \sin. (m+2)\varphi + \delta r^3. \sin. (m+3)\varphi + \&c.$$
$$0 = \alpha. \sin. m\varphi + \beta r. \sin. (m-1)\varphi + \gamma r^2. \sin. (m-2)\varphi + \delta r^3. \sin. (m-3)\varphi + \&c.$$

Sin autem æquatio prior multiplicetur per $\cos. m\varphi$ & posterior per $\sin. m\varphi$, per additionem ac subtractionem sequentes emergent æquationes.

$$0 = \alpha. \cos. m\varphi + \beta r. \cos. (m-1)\varphi + \gamma r^2. \cos. (m-2)\varphi + \delta r^3. \cos. (m-3)\varphi + \&c.$$
$$0 = \alpha. \cos. m\varphi + \beta r. \cos. (m+1)\varphi + \gamma r^2. \cos. (m+2)\varphi + \delta r^3. \cos. (m+3)\varphi + \&c.$$

Hujusmodi ergo duæ æquationes quæcunque conjunctæ determinabunt incognitas r & φ; quod cum plerumque pluribus modis fieri posiit, simul plures Factores trinomiales obtinentur, iique adeo omnes, quos Functio proposita in se complectitur. CAP. IX.

150. Quo usus harum regularum clarius appareat, quarumdam Functionum sæpius occurrentium Factores trinomiales hic indagabimus, ut eos, quoties occasio postulaverit, hinc depromere liceat. Sit itaque proposita hæc Functio $a^n + z^n$, cujus Factores trinomiales formæ $pp - 2pqz . cof. \varphi + qqzz$ determinari oporteat; posito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes: $o = a^n + r^n . cof. n\varphi$ & $o = r^n . fin. n\varphi$, quarum posterior dat $fin. n\varphi = o$; unde erit $n\varphi$ Arcus vel hujus formæ $(2k+1)\pi$ vel $2k\pi$, denotante k numerum integrum. Casus hos ideo distinguo, quod eorum Cosinus sint differentes; priori enim casu erit $cof. (2k+1)\pi = -1$ posteriori casu autem $cof. 2k\pi = +1$. Patet autem priorem formam $n\varphi = (2k+1)\pi$ sumi debere, quippe quæ dat $cof. n\varphi = -1$, unde fit $o = a^n - r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a$, $q = 1$, & $\varphi = \frac{(2k+1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit $aa - 2az . cof. \frac{(2k+1)\pi}{n} + zz$. Cum igitur pro k numerum quemque integrum ponere liceat, prodeunt hoc modo plures Factores, neque tamen infiniti, quoniam si $2k+1$, ultra n augetur, Factores priores recurrunt, quod ex exemplis clarius patebit, cum sit $cof. (2\pi \pm \varphi) = cof. \varphi$. Deinde si n est numerus impar, posito $2k+1 = n$, erit Factor quadratus $aa + 2az + zz$ neque vero hinc sequitur quadratum $(a+z)^2$ esse Factorem Functionis $a^n + z^n$, quoniam (in §. 148) unica æquatio resultat, qua tantum patet $a + z$ esse Divisorem

formulæ $a^n + z^n$; quæ regula ſemper eſt tenenda quoties $cof. \varphi$ fit vel $+1$ vel -1.

Exemplum.

Evolvamus aliquot caſus, quo iſti Factores clarius ob oculos ponatur, atque hos caſus in duas claſſes diſtribuamus, prout n fuerit numerus vel par vel impar.

Si $n = 1$ Formulæ $a + z$ Factor eſt $a + z$	Si $n = 2$ Formulæ $a^2 + z^2$ Factor eſt $a^2 + z^2$
Si $n = 3$ Formulæ $a^3 + z^3$ Factores ſunt $aa - 2az.\ cof. \frac{1}{3}\pi + zz$ $a + z$	Si $n = 4$ Formulæ $a^4 + z^4$ Factores ſunt $aa - 2az.\ cof. \frac{1}{4}\pi + zz$ $aa - 2az.\ cof. \frac{3}{4}\pi + zz$
Si $n = 5$ Formulæ $a^5 + z^5$ Factores ſunt $aa - 2az.\ cof. \frac{1}{5}\pi + zz$ $aa - 2az.\ cof. \frac{3}{5}\pi + zz$ $a + z$	Si $n = 6$ Formulæ $a^6 + z^6$ Factores ſunt $aa - 2az.\ cof. \frac{1}{6}\pi + zz$ $aa - 2az.\ cof. \frac{3}{6}\pi + zz$ $aa - 2az.\ cof. \frac{5}{6}\pi + zz$

Ex quibus exemplis patet omnes Factores obtineri, ſi loco $2k + 1$ omnes numeri impares non majores, quam Exponens n,

n, substituantur, iis vero casibus quibus Factor quadratus prodit, tantum ejus radicem Factoribus annumerari debere.

151. Si proposita sit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz.\ cos.\ \varphi + qqzz$, si posito $r = \frac{p}{q}$, fuerit $o = a^n - r^n.\ cos.\ n\varphi$ & $o = r^n.\ sin.\ n\varphi$. Erit ergo iterum $sin.\ n\varphi = o$, ideoque $n\varphi = (2k+1)\pi$ vel $n\varphi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $cos.\ n\varphi = +1$, qui dat $o = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$, $q = 1$ & $\varphi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propositæ erit $= aa - 2az.\ cos.\ \frac{2k}{n}\pi + zz$; quæ forma, si loco $2k$ omnes numeri pares non majores quam n ponantur, simul dabit omnes Factores; ubi de Factoribus quadratis idem est tenendum quod ante monuimus. Ac primo quidem, posito $k = o$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si n fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formæ $a^n - z^n$.

E X E M P L U M.

Casus Exponentis n ut ante tractati ita se habebunt, prout n fuerit numerus vel impar vel par.

Si $n = 1$ Formulæ $a - z$ ipsa erit Factor $a - z$	Si $n = 2$ Formulæ $a^2 - z^2$ Factores erunt $a - z$ $a + z$
Si $n = 3$ Formulæ $a^3 - z^3$ Factores erunt $a - z$ $aa - 2az.\ cos.\ \frac{2}{3}\pi + zz$	Si $n = 4$ Formulæ $a^4 - z^4$ Factores erunt $a - z$ $aa - 2az.\ cos.\ \frac{2}{4}\pi + zz$ $a + z$
Si $n = 5$ Formulæ $a^5 - z^5$ Factores erunt $a - z$ $aa - 2az.\ cos.\ \frac{2}{5}\pi + zz$ $aa - 2az.\ cos.\ \frac{4}{5}\pi + zz$	Si $n = 6$ Formulæ $a^6 - z^6$ Factores erunt $a - z$ $aa - 2az.\ cos.\ \frac{2}{6}\pi + zz$ $aa - 2az.\ cos.\ \frac{4}{6}\pi + zz$ $a + z$

152. His igitur confirmatur id, quod ſupra jam innuimus, omnem Functionem integram, ſi non in Factores ſimplices reales, tamen in Factores duplices reales reſolvi poſſe. Vidimus enim hanc Functionem indefinitæ dimenſionis $a^n \pm z^n$ ſemper in Factores duplices reales, præter ſimplices reales, reſolvi poſſe. Progrediamur ergo ad Functiones magis compoſitas, uti: $\alpha + \beta z^n + \gamma z^{2n}$, cujus quidem, ſi duos habeat Factores formæ $\eta + \theta z^n$, reſolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $\alpha + \beta z^n + \gamma z^{2n}$, eo caſu, quo non habet duos Factores reales formæ $\eta + \theta z^n$,

resolutionem in Factores reales, vel simplices vel duplices, doceamus. CAP. IX.

153. Consideremus ergo hanc Functionem: $a^{2n} - 2a^n z^n \times cos. g + z^{2n}$, quæ in duos Factores formæ $\eta - \theta z^n$ reales resolvi nequit. Quod si ergo ponamus hujus Functionis Factorem duplicem realem esse $pp - 2pqz. cos. \varphi + qqzz$, posito $r = \frac{p}{q}$, duæ sequentes æquationes erunt resolvendæ: $o = a^{2n} - 2a^n r^n. cos. g. cos. n\varphi + r^{2n}. cos. 2n\varphi$ & $o = - 2a^n r^n. cos. g. sin. n\varphi + r^{2n}. sin. 2n\varphi$. Vel, loco prioris æquationis sumatur, ex §. 149, (ponendo $m = 2n$) hæc $o = a^{2n}. sin. 2n\varphi - 2a^n r^n. cos. g. sin. n\varphi$, quæ cum posteriori collata dat $r = a$; tum vero erit $sin. 2n\varphi = 2 cos. g. sin. n\varphi$. At est $sin. 2n\varphi = 2 sin. n\varphi. cos. n\varphi$, unde fit $cos. n\varphi = cos. g$. At est semper $cos. (2k\pi \pm g) = cos. g$, ex quo habetur $n\varphi = 2k\pi \pm g$ & $\varphi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propositæ erit $= aa - 2az. cos. \frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, si pro $2k$ omnes numeri pares non majores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

E X E M P L U M.

Consideremus ergo casus quibus n est 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erunt ergo

Formulæ

$a^2 - 2az. cos. g + zz$

Unicus Factor

$aa - 2az. cos. g + zz$

Formulæ

$a^4 - 2a^2z^2. cos. g + z^4$

Factores duo

$aa - 2az.\ cos.\ \frac{g}{2} + z^2$

$aa - 2az.\ cos.\ \frac{2\pi + g}{2} + zz$ seu $aa + 2az.\ cos.\ \frac{g}{2} + zz$

Formulæ

$a^4 - 2a^2z^2.\ cos.\ g + z^4$

Factores tres

$aa - 2az.\ cos.\ \frac{g}{3} + z^2$

$aa - 2az.\ cos.\ \frac{2\pi - g}{3} + z^2$

$aa - 2az.\ cos.\ \frac{2\pi + g}{3} + z^2$

Formulæ

$a^6 - 2a^3z^3.\ cos.\ g + z^6$

Factores quatuor

$aa - 2az.\ cos.\ \frac{g}{4} + zz$

$aa - 2az.\ cos.\ \frac{2\pi - g}{4} + zz$

$aa - 2az.\ cos.\ \frac{2\pi + g}{4} + zz$

$aa - 2az.\ cos.\ \frac{4\pi + g}{4} + zz$ seu $aa + 2az.\ cos.\ \frac{g}{4} + zz$

Formulæ

$a^{10} - 2a^5z^5.\ cos.\ g + z^{10}$

Factores quinque

$aa - 2az.\ cos.\ \frac{g}{5} + zz$

$aa - 2az.\ cos.\ \frac{2\pi - g}{5} + zz$

$aa - 2az.\ cos.\ \frac{2\pi + g}{5} + zz$

$aa - 2az.\ cos.\ \frac{4\pi - g}{5} + zz$

$aa - 2az.\ cos.\ \frac{4\pi + g}{5} + zz$

Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, sive simplices sive duplices, resolvi posse. CAP. IX.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel simplices vel duplices, exhiberi possunt; alter vero multiplicator formæ $\iota + \kappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores resolvi poterit. Deinde hæc Functio $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, similiter in Factores, vel simplices vel duplices, reales resolvitur. Quin etiam progredi licet ad formam $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \zeta z^{5n}$, quæ cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio resolutionem in Factores reales, vel simplices vel duplices, admittet. Quare si ullum dubium mansisset circa hujusmodi resolutionem omnium Functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam potest hæc in Factores resolutio ad Series infinitas; scilicet, quia vidimus supra esse $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero esse $e^x = (1 + \frac{x}{i})^i$, denotante i numerum infinitum, perspicuum est Seriem $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos simplices inter se æquales nempe $1 + \frac{x}{i}$. At si ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$

LIB. I.

&c. $= e^x - 1 = (1 + \frac{x}{i})^i - 1$, cujus formæ cum §. 151 comparatæ, quo fit $a = 1 + \frac{x}{i}$, $n = i$ & $z = 1$, Factor quicunque erit $= (1 + \frac{x}{i})^2 - 2(1 + \frac{x}{i})\, cos. \frac{2k}{i}\pi + 1$, unde, substituendo pro $2k$ omnes numeros pares, simul omnes Factores prodibunt. Posito autem $2k = 0$ prodit Factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet, erit ergo x Factor expressionis $e^x - 1$. quod quidem sponte patet. Ad reliquos Factores inveniendos notari oportet esse, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $cos. \frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis sequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^x - 1$ erit divisibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$. Quare expressio $e^x - 1 = x(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.)$, præter Factorem x, habebit hos infinitos $(1 + \frac{x}{i} + \frac{xx}{4\pi\pi})(1 + \frac{x}{i} + \frac{xx}{16\pi\pi})(1 + \frac{x}{i} + \frac{xx}{36\pi\pi})(1 + \frac{x}{i} + \frac{xx}{64\pi\pi})$ &c.

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in singulis insit, atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non potest. Ad hoc ergo incommodum vitandum consideremus hanc expressionem $e^x - e^{-x} = (1 + \frac{x}{i})^i - (1 - \frac{x}{i})^i = 2(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.)$

est enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum §. 151 comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$ unde hujus expressionis Factor erit $= aa - 2az.\, cos. \frac{2k}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right) cos. \frac{2k}{i}\pi = \frac{4xx}{ii} + \frac{4kk}{ii}\pi\pi - \frac{4kk\pi\pi xx}{i^4}$, ob $cos. \frac{2k}{i}\pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x - e^{-x}$ divisibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem terminus $\frac{xx}{ii}$ tuto omittitur, quia etsi per i multiplicetur, tamen manet infinite parvus. Præterea vero ut ante, si $k = 0$, erit primus Factor $= x$. Quocirca, his Factoribus in ordinem redactis, erit $\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \&c. = x\left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.\ldots 7} + \&c.\right)$. Singulis scilicet Factoribus per multiplicationem constantis ejusmodi formam dedi, ut per actualem multiplicationem primus terminus x resultet.

157. Eodem modo cum sit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c. = \frac{\left(1 + \frac{x}{i}\right)^i + \left(1 - \frac{x}{i}\right)^i}{2}$, hujus expressionis cum superiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$, $z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times cos. \frac{2k+1}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right) cos. \frac{2k+1}{i}\pi$. Est

LIB. I. autem $cos. \frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2\pi\pi}{2ii}$, unde forma Factoris erit $= \frac{4xx}{ii} + \frac{(2k+1)^2\pi^2}{ii}$, evanescente termino cujus denominator est i^3. Quoniam ergo omnis Factor expressionis $1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c.$ hujusmodi formam habere debet $1 + \alpha xx$, quo Factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$: hinc Factor formæ propositæ erit $1 + \frac{4xx}{(2k+1)^2\pi\pi}$, ex eoque omnes Factores infiniti invenientur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$
$$\left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponentiales in Sinum & Cosinum cujuspiam Arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit $\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} =$ $fin.\, z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3....7} + \&c.$, quæ adeo expressio hos habet Factores numero infinitos $z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right)$ &c., seu erit $fin.\, z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)$ $\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)$ &c.. Quoties ergo Arcus z ita est comparatus, ut quispiam Factor evanescat, quod fit si $z = 0$, $z = \pm\pi$, $z = \pm 2\pi$, & generaliter si $z = \pm k\pi$, denotante k numerum quemcunque integrum, simul Sinus

nus ejus Arcus debet esse $= 0$, quod quidem ita patet, ut hinc istos Factores a posteriori eruere licuisset. CAP. IX.

Simili modo, cum sit $\frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = cos. z$, erit quoque $cos. z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., seu, his Factoribus in binos resolvendis, erit quoque $cos. z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, si fuerit $z = \pm \frac{(2k+1)}{2}\pi$, fore $cos. z = 0$, id quod etiam ex natura Circuli liquet.

159. Ex §. 152. etiam inveniri possunt Factores hujus expressionis $e^x - 2\,cos. g + e^{-x} = 2(1 - cos. g + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c.)$. Transit enim hæc expressio in hanc $(1 + \frac{x}{i})^i - 2\,cos. g + (1 - \frac{x}{i})^i$, quæ cum illa forma comparata dat $2n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$, unde Factor quicunque hujus formulæ erit $= aa - 2az.\, cos. \frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}).\, cos. \frac{2(2k\pi \pm g)}{i}$: at est $cos. \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, seu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$. Si ergo expressio per $2(1 - cos. g)$ dividatur, ut in Serie infinita terminus constans sit $= 1$, erit, sumendis omnibus Factoribus, $\frac{e^x - 2\,cos. g + e^{-x}}{2(1 - cos. g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

LIB. I. $\left(1+\frac{xx}{(2\pi+g)^2}\right)\left(1+\frac{xx}{(4\pi-g)^2}\right)\left(1+\frac{xx}{(4\pi+g)^2}\right)$ $\left(1+\frac{xx}{(6\pi-g)^2}\right)\left(1+\frac{xx}{(6\pi+g)^2}\right)$ &c.. Atque, si loco x ponatur $z\sqrt{-1}$, erit $\frac{cos.\,z - cos.\,g}{1-cos.\,g} = \left(1-\frac{z}{g}\right)\left(1+\frac{z}{g}\right)$ $\left(1-\frac{z}{2\pi-g}\right)\left(1+\frac{z}{2\pi-g}\right)\left(1-\frac{z}{2\pi+g}\right)\left(1+\frac{z}{2\pi+g}\right)$ $\left(1-\frac{z}{4\pi-g}\right)\left(1+\frac{z}{4\pi-g}\right)$ &c. $= 1 - \frac{zz}{1.2\,(1-cos.\,g)} + \frac{z^4}{1.2.3.4\,(1-cos.\,g)} - \frac{z^6}{1.2\ldots.6\,(1-cos.\,g)} +$ &c.. Hujus adeo Seriei in infinitum continuatæ Factores omnes cognoscuntur.

160. Commode etiam hujusmodi Functionis $e^{b+x} \pm e^{c-x}$ Factores inveniri omnesque assignari possunt. Transmutatur enim in hanc formam $\left(1+\frac{b+x}{i}\right)^i \pm \left(1+\frac{c-x}{i}\right)^i$, quæ comparata cum forma $a^i \pm z^i$, Factorem habebit $aa - 2az.\,cos.\frac{m\pi}{i} + zz$, denotante m numerum imparem si valeat signum superius, contra vero numerum parem. Cum autem, ob i numerum infinite magnum, fit $cos.\frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii}$, erit Factor ille generalis $= (a-z)^2 + \frac{mm\pi\pi}{ii}az$. At hoc casu erit $a = 1 + \frac{b+x}{i}$ & $z = 1 + \frac{c-x}{i}$, unde fit $(a-z)^2 = \frac{(b-c+2x)^2}{ii}$ & $az = 1 + \frac{b-c}{i} + \frac{bc+(c-b)x-xx}{ii}$; ideoque Factor erit per ii multiplicatus $= (b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$, neglectis terminis per i vel ii divisis, quoniam jam omnis generis termini adsunt, præ quibus hi evanescerent. Termino ergo constante ad unitatem per divisionem reducto erit Factor $= 1 + \frac{4(b-c)x+4xx}{mm\pi\pi+(b-c)^2}$.

161. Nunc, quoniam in omnibus Factoribus terminus constans est $= 1$, ipsa Functio $e^{b+x} \pm e^{c-x}$ per ejusmodi constantem dividi debet, ut terminus constans fiat $= 1$, seu ut ejus valor, posito $x = 0$, fiat $= 1$; talis Divisor erit $e^b \pm e^c$, & hanc ob rem expressio hæc $\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per Factores numero infinitos exponi poterit. Erit ergo, si valeat signum superius atque m denotet numerum imparem, $\frac{e^{b+x} + e^{c-x}}{e^b + e^c} = (1 + \frac{4(b-c)x + 4xx}{\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{9\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{25\pi\pi + (b-c)^2})$ &c.; sin autem signum inferius valeat, atque ideo m denotet numerum parem, casuque $m = 0$ radix Factoris quadrati ponatur, erit $\frac{e^{b+x} - e^{c-x}}{e^b - e^c} = (1 + \frac{2x}{b-c})(1 + \frac{4(b-c)x + 4xx}{4\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{16\pi\pi + (b-c)^2})(1 + \frac{4(b-c)x + 4xx}{36\pi\pi + (b-c)^2})$ &c.

162. Ponatur $b = 0$, quod sine detrimento universalitatis fieri potest, eritque $\frac{e^x + e^c e^{-x}}{1 + e^c} = (1 - \frac{4cx + 4xx}{\pi\pi + cc})(1 - \frac{4cx + 4xx}{9\pi\pi + cc})(1 - \frac{4cx + 4xx}{25\pi\pi + cc})$ &c.; $\frac{e^x - e^c e^{-x}}{1 - e^c} = (1 - \frac{2x}{c})(1 - \frac{4cx + 4xx}{4\pi\pi + cc})(1 - \frac{4cx + 4xx}{16\pi\pi + cc})(1 - \frac{4cx + 4xx}{36\pi\pi + cc})$ &c.. Jam ponatur c negativum, atque habebuntur hæ duæ æquationes: $\frac{e^x + e^{-c} e^{-x}}{1 + e^{-c}} =$

LIB. I. $\left(1+\frac{4cx+4xx}{\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{9\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{25\pi\pi+cc}\right)$ &c.,

$$\frac{e^{x}-e^{-c}e^{-x}}{1-e^{-c}}=\left(1+\frac{2x}{c}\right)\left(1+\frac{4cx+4xx}{4\pi\pi+cc}\right)$$

$\left(1+\frac{4cx+4xx}{16\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{36\pi\pi+cc}\right)$ &c.. Multiplicetur forma prima per tertiam, ac prodibit $\frac{e^{2x}+e^{-2x}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}}$; ponatur vero y loco $2x$, eritque $\frac{e^{y}+e^{-y}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}}=\left(1-\frac{2cy+yy}{\pi\pi+cc}\right)$
$\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{25\pi\pi+cc}\right)$
$\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)$ &c.. Multiplicetur prima forma per quartam, erit productum $=\frac{e^{2x}-e^{-2x}+e^{c}-e^{-c}}{e^{c}-e^{-c}}$; ponatur y pro $2x$, eritque $\frac{e^{y}-e^{-y}+e^{c}-e^{-c}}{e^{c}-e^{-c}}=$
$\left(1+\frac{y}{c}\right)\left(1-\frac{2cy+yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{9\pi\pi+cc}\right)$
$\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{25\pi\pi+cc}\right)$ &c.. Si secunda forma per tertiam multiplicetur, prodibit eadem æquatio nisi quod c capiendum sit negativum, erit nempe

$$\frac{e^{c}-e^{-c}-e^{y}+e^{-y}}{e^{c}-e^{-c}}=\left(1-\frac{y}{c}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)$$

$\left(1-\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{16\pi\pi+cc}\right)$
$\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{36\pi\pi+cc}\right)$ &c.. Multiplicetur de-

nique forma secunda per quartam eritque $\frac{e^y+e^{-y}-e^c-e^{-c}}{2-e^c-e^{-c}}$ $=(1-\frac{yy}{cc})(1-\frac{2cy+yy}{4\pi\pi+cc})(1+\frac{2cy+yy}{4\pi\pi+cc})(1-\frac{2cy+yy}{16\pi\pi+cc})$ $(1+\frac{2cy+yy}{16\pi\pi+cc})(1-\frac{2cy+yy}{36\pi\pi+cc})(1+\frac{2cy+yy}{36\pi\pi+cc})$ &c.

163. Hæ quatuor combinationes nunc commode ad Circulum transferri possunt, ponendo $c=g\surd-1$ & $y=v\surd-1$: erit enim $e^{v\surd-1}+e^{-v\surd-1}=2\,\cos.\,v$; $e^{v\surd-1}-e^{-v\surd-1}=2\surd-1.\sin.\,v.$ & $e^{g\surd-1}+e^{-g\surd-1}=2\cos.\,g$; $e^{g\surd-1}-e^{-g\surd-1}=2\surd-1.\sin.\,g.$ Hinc prima combinatio dabit $\frac{\cos.\,v+\cos.\,g}{1+\cos.\,g}=1-\frac{vv}{1.2(1+\cos.\,g)}+\frac{v^4}{1.2.3.4(1+\cos.\,g)}-\frac{v^6}{1.2\ldots6(1+\cos.\,g)}+\&c.=(1+\frac{2gv-vv}{\pi\pi-gg})$ $(1-\frac{2gv-vv}{\pi\pi-gg})(1+\frac{2gv-vv}{9\pi\pi-gg})(1-\frac{2gv-vv}{9\pi\pi-gg})$ $(1+\frac{2gv-vv}{25\pi\pi-gg})(1-\frac{2gv-vv}{25\pi\pi-gg})\,\&c.=(1+\frac{v}{\pi-g})$ $(1-\frac{v}{\pi+g})(1-\frac{v}{\pi-g})(1+\frac{v}{\pi+g})(1+\frac{v}{3\pi-g})$ $(1-\frac{v}{3\pi+g})(1-\frac{v}{3\pi-g})(1+\frac{v}{3\pi+g})\,\&c.=$ $(1-\frac{vv}{(\pi-g)^2})(1-\frac{vv}{(\pi+g)^2})(1-\frac{vv}{(3\pi-g)^2})$ $(1-\frac{vv}{(3\pi+g)^2})(1-\frac{vv}{(5\pi-g)^2})$ &c.. Quarta vero combinatio dat $\frac{\cos.\,v-\cos.\,g}{1-\cos.\,g}=1-\frac{vv}{1.2(1-\cos.\,g)}+\frac{v^4}{1.2.3.4(1-\cos.\,g.)}-\frac{v^6}{1.2\ldots6(1-\cos.\,g)}+\&c.=(1-\frac{vv}{gg})$ $(1+\frac{2gv-vv}{4\pi\pi-gg})(1-\frac{2gv-vv}{4\pi\pi-gg})(1+\frac{2gv-vv}{16\pi\pi-gg})$

 $(1 - \frac{2gv - vv}{16\pi\pi - gg})$ &c. $= (1 - \frac{v}{g})(1 + \frac{v}{g})(1 + \frac{v}{2\pi - g})$ $(1 - \frac{v}{2\pi + g})(1 - \frac{v}{2\pi - g})(1 + \frac{v}{2\pi + g})(1 + \frac{v}{4\pi - g})$ $(1 - \frac{v}{4\pi + g})$ &c. $= (1 - \frac{vv}{gg})(1 - \frac{vv}{(2\pi - g)^2})$ $(1 - \frac{vv}{(2\pi + g)^2})(1 - \frac{vv}{(4\pi - g)^2})(1 - \frac{vv}{(4\pi + g)^2})$ &c.. Secunda combinatio dat $\frac{\textit{fin.}\, g + \textit{fin.}\, v}{\textit{fin.}\, g} = 1 + \frac{v}{\textit{fin.}\, g} - \frac{v^3}{1.2.3\,\textit{fin.}\, g} + \frac{v^5}{1.2.\ldots 5\,\textit{fin.}\, g}$ — &c. $= (1 + \frac{v}{g})(1 + \frac{2gv - vv}{\pi\pi - gg})$ $(1 - \frac{2gv - vv}{4\pi\pi - gg})(1 + \frac{2gv - vv}{9\pi\pi - gg})(1 + \frac{2gv - vv}{16\pi\pi - gg})$ &c. $= (1 + \frac{v}{g})(1 + \frac{v}{\pi - g})(1 - \frac{v}{\pi + g})(1 - \frac{v}{2\pi - g})$ $(1 + \frac{v}{2\pi + g})(1 + \frac{v}{3\pi - g})(1 - \frac{v}{3\pi + g})(1 - \frac{v}{5\pi - g})$ &c.. Ac fumto v negativo prodit tertia combinatio.

164. Ipfæ vero etiam expreffiones in §. 162 primum inventæ ad Arcus circulares traduci poffunt hoc modo: cum fit $\frac{e^x + e^c e^{-x}}{1 + e^c} = \frac{(1 + e^{-c})(e^x + e^c e^{-x})}{2 + e^c + e^{-c}} =$ $\frac{e^x + e^{-x} + e^{c - x} + e^{-c + x}}{2 + e^c + e^{-c}}$, fi ponamus $c = g\surd -1$ & $x = z\surd -1$, hæc expreffio abit in hanc $\frac{\textit{cof.}\, z + \textit{cof.}\, (g - z)}{1 + \textit{cof.}\, g} =$ $\textit{cof.}\, z + \frac{\textit{fin.}\, g.\ \textit{fin.}\, z}{1 + \textit{cof.}\, g}$. Erit ergo (ob $\frac{\textit{fin.}\, g}{1 + \textit{cof.}\, g} = \textit{tang.}\, \frac{1}{2} g$) $\textit{cof.}\, z + \textit{tang.}\, \frac{1}{2} g.\ \textit{fin.}\, z = 1 + \frac{z}{1}\ \textit{tang.}\, \frac{1}{2} g - \frac{zz}{1.2} -$ $\frac{z^3}{1.2.3}\ \textit{tang.}\, \frac{1}{2} g + \frac{z^4}{1.2.3.4} + \frac{z^5}{1.2.3.4.5}\ \textit{tang.}\, \frac{1}{2} g$ — &c. $= (1 + \frac{4gz - 4zz}{\pi\pi - gg})(1 + \frac{4gz - 4zz}{9\pi\pi - gg})(1 + \frac{4gz - 4zz}{25\pi\pi - gg})$ &c.

$= (1 + \frac{2z}{\pi - g})(1 - \frac{2z}{\pi + g})(1 + \frac{2z}{3\pi - g})(1 - \frac{2z}{3\pi + g})$ $(1 + \frac{2z}{5\pi - g})(1 - \frac{2z}{5\pi + g})$ &c.. Simili modo altera expressio, si Numerator & Denominator per $1 - e^{-c}$ multiplicetur, abit in $\frac{e^x + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^c - e^{-c}}$; quæ, facto $c = g\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\frac{\text{cos.}\, z - \text{cos.}\,(g - z)}{1 - \text{cos.}\, g} = \text{cos.}\, z - \frac{\text{sin.}\, g.\ \text{sin.}\, z}{1 - \text{cos.}\, g} = \text{cos.}\, z - \frac{\text{sin.}\, z}{\text{tang.}\, \frac{1}{2} g}$. Erit ergo $\text{cos.}\, z - \text{cot.}\, \frac{1}{2} g.\ \text{sin.}\, z = 1 - \frac{z}{1}\, \text{cot.}\, \frac{1}{2} g - \frac{zz}{1.2} + \frac{z^3}{1.2.3}\, \text{cot.}\, \frac{1}{2} g + \frac{z^4}{1.2.3.4} - \frac{z^5}{1.2\ldots5}\, \text{cot.}\, \frac{1}{2} g + \&c. = (1 - \frac{2z}{g})(1 + \frac{4gz - 4zz}{4\pi\pi - gg})$ $(1 + \frac{4gz - 4zz}{16\pi\pi - gg})(1 + \frac{4gz - 4zz}{36\pi\pi - gg})$ &c. $= (1 - \frac{2z}{g})$ $(1 + \frac{2z}{2\pi - g})(1 - \frac{2z}{2\pi + g})(1 + \frac{2z}{4\pi - g})(1 - \frac{2z}{4\pi + g})$ &c.

Quod si ergo ponatur $v = 2z$ seu $z = \frac{1}{2} v$; habebitur

$\frac{\text{cos.}\, \frac{1}{2}(g - v)}{\text{cos.}\, \frac{1}{2} g} = \text{cos.}\, \frac{1}{2} v + \text{tang.}\, \frac{1}{2} g.\ \text{sin.}\, \frac{1}{2} v = (1 + \frac{v}{\pi - g})(1 - \frac{v}{\pi + g})(1 + \frac{v}{3\pi - g})(1 - \frac{v}{3\pi + g})$ &c.;

$\frac{\text{cos.}\, \frac{1}{2}(g + v)}{\text{cos.}\, \frac{1}{2} g} = \text{cos.}\, \frac{1}{2} v - \text{tang.}\, \frac{1}{2} g.\ \text{sin.}\, \frac{1}{2} v = (1 - \frac{v}{\pi - g})(1 + \frac{v}{\pi + g})(1 - \frac{v}{3\pi - g})(1 + \frac{v}{3\pi + g})$ &c.;

$\frac{\text{sin.}\, \frac{1}{2}(g - v)}{\text{sin.}\, \frac{1}{2} g} = \text{cos.}\, \frac{1}{2} v - \text{cot.}\, \frac{1}{2} g.\ \text{sin.}\, \frac{1}{2} v = (1 - \frac{v}{g})(1 + \frac{v}{2\pi - g})(1 - \frac{v}{2\pi + g})(1 + \frac{v}{4\pi - g})$ &c.;

$$\frac{fin. \frac{1}{2}(g+v)}{fin. \frac{1}{2} g} = cof. \tfrac{1}{2} v + cot. \tfrac{1}{2} g. \; fin. \tfrac{1}{2} v = (1 + \tfrac{v}{g})(1 - \tfrac{v}{2\pi - g})(1 + \tfrac{v}{2\pi + g})(1 - \tfrac{v}{4\pi - g}) \text{ \&c.}$$

Quorum Factorum lex progreſſionis ſatis eſt ſimplex & uniformis; atque ex his expreſſionibus per multiplicationem oriuntur eæ ipſæ, quæ in §. præcedente ſunt inventæ.

CAPUT X.

De uſu Factorum inventorum in definiendis ſummis Serierum infinitarum.

165. Si fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 +$ &c. $= (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)$ &c., hi Factores, ſive ſint numero finiti ſive infiniti, ſi in ſe actu multiplicentur, illam expreſſionem $1 + Az + Bz^2 + Cz^3 + Dz^4 +$ &c. producere debent. Æquabitur ergo coëfficiens A ſummæ omnium quantitatum $\alpha + \beta + \gamma + \delta + \epsilon +$ &c.. Coëfficiens vero B æqualis erit ſummæ productorum ex binis, eritque $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta +$ &c.. Tum vero coëfficiens C æquabitur ſummæ productorum ex ternis, nempe erit $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta +$ &c.. Atque ita porro erit $D =$ ſummæ productorum ex quaternis, $E =$ ſummæ productorum ex quinis, &c., id quod ex Algebra communi conſtat.

166. Quia ſumma quantitatum $\alpha + \beta + \gamma + \delta +$ &c., datur una cum ſumma productorum ex binis, hinc ſumma Quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 +$ &c., inveniri poterit, quippe quæ æqualis eſt Quadrato ſummæ demtis duplicibus productis ex binis. Simili modo ſumma Cuborum, Biquadratorum & altiorum Poteſtatum definiri poteſt: ſi enim ponamus

$P =$

$$\begin{aligned}
P &= \alpha + \beta + \gamma + \delta + \varepsilon + \&c.\\
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c.\\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \&c.\\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \&c.\\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \&c.\\
V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \&c.\\
&\&c.
\end{aligned}$$

Valores P, Q, R, S, T, V &c. fequenti modo ex cognitis A, B, C, D, &c. determinabuntur.

$$\begin{aligned}
P &= A\\
Q &= AP - 2B\\
R &= AQ - BP + 3C\\
S &= AR - BQ + CP - 4D\\
T &= AS - BR + CQ - DP + 5E\\
V &= AT - BS + CR - DQ + EP - 6F\\
&\&c.
\end{aligned}$$

quarum formularum veritas examine inftituto facile agnofcitur.

167. Cum igitur fupra (§. 156) invenerimus effe: $\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2....7} + \&c.\right) = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right)$ &c., erit $1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3....7} + \&c. = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)$ &c. Ponatur $xx = \pi\pi z$, eritque $1 + \frac{\pi\pi}{1.2.3} z + \frac{\pi^4}{1.2.3.4.5} z^2 + \frac{\pi^6}{1.2.3....7} z^3 + \&c. = (1+z)\left(1+\frac{1}{4}z\right)\left(1+\frac{1}{9}z\right)\left(1+\frac{1}{16}z\right)$

 $(1+\frac{1}{25}\zeta)$ &c.. Facta ergo applicatione superioris regulæ ad hunc casum, erit $A=\frac{\pi\pi}{6}$; $B=\frac{\pi^4}{120}$; $C=\frac{\pi^6}{5040}$; $D=\frac{\pi^8}{362880}$ &c.. Quod si ergo ponatur

$$P = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\&c.$$
$$Q = 1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\frac{1}{36^2}+\&c.$$
$$R = 1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\frac{1}{36^3}+\&c.$$
$$S = 1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\frac{1}{36^4}+\&c.$$
$$T = 1+\frac{1}{4^5}+\frac{1}{9^5}+\frac{1}{16^5}+\frac{1}{25^5}+\frac{1}{36^5}+\&c.$$
&c.

atque harum litterarum valores ex *A*, *B*, *C*, *D*, &c. determinentur, prodibit.

$$P = \frac{\pi\pi}{6}$$
$$Q = \frac{\pi^4}{90}$$
$$R = \frac{\pi^6}{945}$$
$$S = \frac{\pi^8}{9450}$$
$$T = \frac{\pi^{10}}{93555}$$
&c.

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+$ &c., contentarum, quoties n fuerit numerus par, summam ope Peripheriæ Circuli π exhiberi posse; habebit enim semper summa Seriei ad π^n rationem

rationalem. Quo autem valor harum ſummarum clarius perſpiciatur, plures hujuſmodi Serierum ſummas commodiori modo expreſſas hic adjiciam.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\frac{2^0}{1.2.3}.\frac{1}{1}\pi^2$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.=\frac{2^2}{1.2.3.4.5}.\frac{1}{3}\pi^4$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c.=\frac{2^4}{1.2.3\ldots7}.\frac{1}{3}\pi^6$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c.=\frac{2^6}{1.2.3\ldots9}.\frac{3}{5}\pi^8$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\&c.=\frac{2^8}{1.2.3\ldots11}.\frac{5}{3}\pi^{10}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\&c.=\frac{2^{10}}{1.2.3\ldots13}.\frac{691}{105}\pi^{12}$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\&c.=\frac{2^{12}}{1.2.3\ldots15}.\frac{35}{1}\pi^{14}$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\&c.=\frac{2^{14}}{1.2.3\ldots17}.\frac{3617}{15}\pi^{16}$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\&c.=\frac{2^{16}}{1.2.3\ldots19}.\frac{43867}{21}\pi^{18}$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\&c.=\frac{2^{18}}{1.2.3\ldots21}.\frac{1222277}{55}\pi^{20}$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\&c.=\frac{2^{20}}{1.2.3\ldots23}.\frac{854513}{3}\pi^{22}$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\&c.=\frac{2^{22}}{1.2.3\ldots\ldots\ldots25}.\frac{1181820455}{273}\pi^{24}$$

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\&c.=\frac{2^{24}}{1.2.3\ldots\ldots\ldots27}.\frac{76977927}{1}\pi^{26}.$$

Hucuſque iſtos Poteſtatum ipſius π Exponentes artificio alibi exponendo continuare licuit, quod ideo hic adjunxi, quod

LIB. I. Seriei fractionum primo intuitu perquam irregularis 1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$, $\frac{691}{105}$, $\frac{35}{1}$, &c. in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo æquationem §. 157 inventam, ubi erat $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} +$ &c. $= (1 + \frac{4xx}{\pi\pi})(1 + \frac{4xx}{9\pi\pi})(1 + \frac{4xx}{25\pi\pi})(1 + \frac{4xx}{49\pi\pi})$ &c.. Posito ergo $xx = \frac{\pi\pi z}{4}$ erit $1 + \frac{\pi\pi}{1.2.4} z + \frac{\pi^4}{1.2.3.4.4^2} zz + \frac{\pi^6}{1.2.\ldots 6.4^3} z^3 +$ &c. $= (1 + z)(1 + \frac{1}{9} z)(1 + \frac{1}{25} z)(1 + \frac{1}{49} z)$ &c.. Unde, facta applicatione, erit $A = \frac{\pi\pi}{1.2.4}$; $B = \frac{\pi^4}{1.2.3.4.4^2}$; $C = \frac{\pi^6}{1.2.3.\ldots 6.4^3}$; &c.. Quod si ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{\&c.}$$
$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \text{\&c.}$$
$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \text{\&c.}$$
$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \text{\&c.}$$
&c.

reperientur sequentes pro P, Q, R, S, &c., valores:

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3}; \qquad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$
$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7}; \qquad S = \frac{272}{1.2.3\ldots 7} \cdot \frac{\pi^8}{2^9}$$

$T = \frac{7936}{1.2.3....9} \cdot \frac{\pi^{10}}{2^{11}}$; $V = \frac{353792}{1.2.3....11} \cdot \frac{\pi^{12}}{2^{13}}$

$W = \frac{22368256}{1.2.3....13} \cdot \frac{\pi^{14}}{2^{15}}$.

170. Eædem ſummæ Poteſtatum numerorum imparium inveniri poſſunt ex ſummis præcedentibus, in quibus omnes numeri occurrunt; ſi enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} +$ &c., erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c., quæ Series numeros tantum pares continens, ſi a priori ſubtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} +$ &c. Quod ſi autem Series $\frac{M}{2^n}$ bis ſumta ſubtrahatur ab M ſigna prodibunt alternantia, eritque $M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} +$ &c.. Per tradita ergo præcepta ſummari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{\&c.}$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

Si quidem n ſit numerus par, atque ſumma erit $= A\pi^n$ exiſtente A numero rationali.

171. Præterea vero expreſſiones §. 164 exhibitæ ſimili modo

 Series notatu dignas suppeditabunt. Cum enim sit $cof. \frac{1}{2} v + tang. \frac{1}{2} g. \, fin. \frac{1}{2} v = (1 + \frac{v}{\pi - g})(1 - \frac{v}{\pi + g})(1 + \frac{v}{3\pi - g})$ &c., si ponamus $v = \frac{x}{n}\pi$ & $g = \frac{m}{n}\pi$ erit $(1 + \frac{x}{n - m})(1 - \frac{x}{n + m})(1 + \frac{x}{3n - m})(1 - \frac{x}{3n + m})(1 + \frac{x}{5n - m})(1 - \frac{x}{5n + m})$ &c. $= cof. \frac{x\pi}{2n} + tang. \frac{m\pi}{2n}. \, fin. \frac{x\pi}{2n} = 1 + \frac{\pi x}{2n} \, tang. \frac{m\pi}{2n} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3 x^3}{2.4.6n^3} \, tang. \frac{m\pi}{2n} + \frac{\pi^4 x^4}{2.4.6.8n^4}$ + &c. Hæc expressio infinita cum §. 165 collata dabit hos valores $A = \frac{\pi}{2n}. tang. \frac{m\pi}{2n}$; $B = \frac{-\pi\pi}{2.4nn}$; $C = \frac{-\pi^3}{2.4.6n^3}. \, tang. \frac{m\pi}{2n}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5}. \, tang. \frac{m\pi}{2n}$ &c. Tum vero erit $\alpha = \frac{1}{n - m}$; $\beta = -\frac{1}{n + m}$; $\gamma = \frac{1}{3n - m}$; $\delta = -\frac{1}{3n + m}$; $\epsilon = \frac{1}{5n - m}$; $\zeta = -\frac{1}{5n + m}$ &c.

172. Hinc ergo ad normam §. 166 sequentes Series exorientur.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \&c.$$

$$Q = \frac{1}{(n - m)^2} + \frac{1}{(n + m)^2} + \frac{1}{(3n - m)^2} + \frac{1}{(3n + m)^2} + \frac{1}{(5n - m)^2} + \&c.$$

$$R = \frac{1}{(n - m)^3} - \frac{1}{(n + m)^3} + \frac{1}{(3n - m)^3} - \frac{1}{(3n + m)^3} + \frac{1}{(5n - m)^3} - \&c.$$

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \&c.$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \&c.$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \&c.$$

&c.

Posito autem *tang.* $\frac{m\pi}{2n} = k$ erit, uti ostendimus,

$$P = A = \frac{k\pi}{2n}$$

$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2.4nn}$$

$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2.4.6n^3}$$

$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2.4.6.8n^4}$$

$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2.4.6.8.10n^5}$$

&c.

173. Pari modo ultima forma §. 164, $cof.\frac{1}{2}v + cot.\frac{1}{2}g \times fin.\frac{1}{2}v = (1+\frac{v}{g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1-\frac{v}{4\pi-g})(1+\frac{v}{4\pi+g})$ &c., si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$ & $tang.\frac{m\pi}{2n} = k$, ut sit $cot.\frac{1}{2}g = \frac{1}{k}$, dabit $cof.\frac{\pi x}{2n} + \frac{1}{k} \times fin.\frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3x^3}{2.4.6n^3k} + \frac{\pi^4x^4}{2.4.6.8n^4} + \frac{\pi^5x^5}{2.4.6.8.10n^5k} - \&c. = (1+\frac{x}{m})(1-\frac{x}{2n-m})(1+\frac{x}{2n+m})$

LIB. I. $(1 - \frac{x}{4n - m})(1 + \frac{x}{4n + m},)$ &c.. Comparatione ergo cum forma generali (§. 165) instituta erit $A = \frac{\pi}{2nk}$; $B = \frac{-\pi\pi}{2.4n^2}$; $C = \frac{-\pi^3}{2.4.6n^3k}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5k}$; &c.; ex Factoribus vero habebitur $\alpha = \frac{1}{m}$; $\beta = \frac{-1}{2n - m}$; $\gamma = \frac{1}{2n + m}$; $\delta = \frac{-1}{4n - m}$; $\epsilon = \frac{1}{4n + m}$ &c.

174. Hinc ergo ad normam §. 166 sequentes Series formabuntur, earumque summæ assignabuntur

$$P = \frac{1}{m} - \frac{1}{2n - m} + \frac{1}{2n + m} - \frac{1}{4n - m} + \frac{1}{4n + m} - \text{\&c.}$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n - m)^2} + \frac{1}{(2n + m)^2} + \frac{1}{(4n - m)^2} + \frac{1}{(4n + m)^2} + \text{\&c.}$$

$$R = \frac{1}{m^3} - \frac{1}{(2n - m)^3} + \frac{1}{(2n + m)^3} - \frac{1}{(4n - m)^3} + \frac{1}{(4n + m)^3} - \text{\&c.}$$

$$S = \frac{1}{m^4} + \frac{1}{(2n - m)^4} + \frac{1}{(2n + m)^4} + \frac{1}{(4n - m)^4} + \frac{1}{(4n + m)^4} + \text{\&c.}$$

$$T = \frac{1}{m^5} - \frac{1}{(2n - m)^5} + \frac{1}{(2n + m)^5} - \frac{1}{(4n - m)} + \frac{1}{(4n + m)^5} - \text{\&c.}$$

&c.

Hæ

Hæ autem ſummæ P, Q, R, S, &c. ita ſe habebunt CAP. X.

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk}$$

$$Q = \frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2.4n^2k^2}$$

$$R = \frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2.4.6n^3k^3}$$

$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+3k^4)\pi^4}{2.4.6.8n^4k^4}$$

$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2.4.6.8.10n^5k^5}$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2.4.6.8.10.12n^6k^6}$$

&c.

175. Series iſtæ generales merentur ut caſus quoſdam particulares inde derivemus, qui prodibunt ſi rationem m ad n in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = tang.\ \frac{\pi}{4} = tang.\ 45^\circ = 1$, atque ambæ Serierum claſſes inter ſe congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$\frac{5\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

&c.

Harum Serierum primam jam ſupra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169) ſunt erutæ; ceteræ, in quibus Exponentes ſunt numeri impa-

LIB. I. res, hic primum occurrunt. Constat ergo omnium quoque istarum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

summas per valorem ipsius π assignari posse.

176. Sit nunc $m = 1$, $n = 3$; erit $k = tang. \frac{\pi}{6} = tang.\ 30^\circ = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c.$$

$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c.$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c.$$

&c., sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c.$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c.$$

&c.

in his Seriebus desunt omnes numeri per ternarium divisibiles: hinc pares dimensiones ex jam inventis deducentur hoc modo. Cum sit

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c., \text{ erit}$$

$$\frac{\pi\pi}{6.9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54},$$

quæ posterior Series continens omnes numeros per ternarium divisibiles, si subtrahatur a priore, remanebunt omnes numeri non divisibiles per 3: sicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$, uti invenimus.

177. Eadem hypotheſis $m = 1$, $n = 3$ & $k = \frac{1}{\sqrt{3}}$, Cap. X. ad §. 174 accommodata has præbebit ſummationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \&c.$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$
$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \&c.$$
&c.

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium ſunt diviſibiles. Ceterum pares dimenſiones ex jam cognitis deduci poſſunt, cum enim ſit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c., \text{ erit}$$
$$\frac{\pi\pi}{8.\,9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c. = \frac{\pi\pi}{72}$$

quæ Series, omnes numeros impares per 3 diviſibiles continens, ſi ſubtrahatur a ſuperiore, relinquet Seriem quadratorum numerorum imparium per 3 non diviſibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

178. Si Series in §. §. 172 & 174 inventæ vel addantur vel ſubtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit ſcilicet $\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \&c. = \frac{(kk+1)\pi}{2nk}$: at eſt $k = \text{tang.}\,\frac{m\pi}{2n} = \frac{\sin.\,\frac{m\pi}{2n}}{\cos.\,\frac{m\pi}{2n}}$, & $1 + kk = \frac{1}{(\cos.\,\frac{m\pi}{2n})^2}$, unde $\frac{2k}{1+kk} = 2\sin.\,\frac{m\pi}{2n} \times \cos.\,\frac{m\pi}{2n} = \sin.\,\frac{m\pi}{n}$, quo valore ſubſtituto habebimus

$$\frac{\pi}{n\sin.\,\frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} +$$

LIB. I. $\frac{1}{3n-m} - \frac{1}{3n+m}$ — &c. . Simili modo per subtractionem erit $\frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-kk)\pi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m}$ — &c. , at est $\frac{2k}{1-kk} = tang.\ 2.\ \frac{m\pi}{2n} = tang.\ \frac{m\pi}{n} = \frac{fin.\ \frac{m\pi}{n}}{cof.\ \frac{m\pi}{n}}$, hinc erit $\frac{\pi . cof. \frac{m\pi}{n}}{n . fin.\ \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m}$ + &c.. Series Quadratorum & altiorum Potestatum hinc ortæ facilius per differentiationem hinc deducentur infra.

179. Quoniam casus, quibus $m = 1$ & $n = 2$ vel 3, jam evolvimus, ponamus $m = 1$ & $n = 4$; erit $fin.\ \frac{m\pi}{n} = fin.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & $cof.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hinc itaque habebitur
$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}$ + &c.
& $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15}$ + &c.. Sit $m = 1$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{\pi}{8}$ & $fin.\ \frac{\pi}{8} = \sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$ & $cof.\ \frac{\pi}{8} = \sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$ & $\frac{cof.\ \frac{\pi}{8}}{fin.\ \frac{\pi}{8}} = 1 + \sqrt{2}$. Hinc itaque erit (178) per primam combinationem
$\frac{\pi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23}$ — &c..

& per secundam, quoniam $1 + \sqrt{2} = \frac{1}{-1+\sqrt{2}}$,

$$\frac{\pi}{8(-1+\sqrt{2})} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \&c.$$

Sit nunc $m = 3$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{3\pi}{8}$ & $\mathit{sin.}\,\frac{3\pi}{8} = \sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$, & $\mathit{cos.}\,\frac{3\pi}{8} = \sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$, unde $\frac{\mathit{cos.}\,\frac{3}{8}\pi}{\mathit{sin.}\,\frac{3}{8}\pi} = \frac{1}{1+\sqrt{2}}$; ac prodibunt hæ Series

$$\frac{\pi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \&c.$$

$$\frac{\pi}{8(1+\sqrt{2})} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \&c.$$

180. Ex his seriebus nascuntur,

1°. Addendo primam tertiæ

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

2°. Subtrahendo tertiam a prima

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

3°. Addendo quartæ primam

$$\frac{\pi(-1+\sqrt{2}+\sqrt{(4+2\sqrt{2})})}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

4°. Subtrahendo quartam a prima

$$\frac{\pi(1-\sqrt{2}+\sqrt{(4+2\sqrt{2})})}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

5°. Addendo fecundam tertiæ

$$\frac{\pi(1+\sqrt{2}+\sqrt{(4-2\sqrt{2})})}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

6°. Subtrahendo tertiam a fecunda

$$\frac{\pi(1+\sqrt{2}-\sqrt{(4-2\sqrt{2})})}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo fummæ reperientur Serierum $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c., in quibus fignorum $+$ & $-$ viciffitudines alias leges fequantur.

181. Si in Seriebus §. 178 inventis bini termini in unam fummam colligantur, erit

$$\frac{\pi}{n.\,fin.\,\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn - mm} - \frac{2m}{4nn - mm} + \frac{2m}{9nn - mm} - \frac{2m}{16nn - mm} + \&c.$$

ideoque

$$\frac{1}{nn - mm} - \frac{1}{4nn - mm} + \frac{1}{9nn - mm} - \&c. = \frac{\pi}{2mn.\,fin.\,\frac{m\pi}{n}} - \frac{1}{2mm}.$$

Altera vero Series dabit

$$\frac{\pi}{n.\,tang.\,\frac{m}{n}\pi} = \frac{1}{m} - \frac{2m}{nn - mm} - \frac{2m}{4nn - mm} - \frac{2m}{9nn - mm} - \&c.$$

hincque

$$\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn.\ tang.\ \frac{m}{n}\pi}.$$

Ex his autem conjunctis nascitur hæc

$$\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\pi.\ tang.\ \frac{m}{2n}\pi}{4mn}.$$

Si in hac Serie sit $n = 1$ & m numerus par quicunque $= 2k$, ob *tang.* $k\pi = 0$, erit semper, nisi sit $k = 0$,

$$\frac{1}{1 - 4kk} + \frac{1}{9 - 4kk} + \frac{1}{25 - 4kk} + \frac{1}{49 - 4kk} + \&c. = 0,$$

sin autem in illa Serie fiat $n = 2$ & m fuerit numerus quicunque impar $= 2k + 1$, ob $\frac{1}{tang.\ \frac{m\pi}{n}} = 0$, erit $\frac{1}{4 - (2k+1)^2} +$

$$\frac{1}{16 - (2k+1)^2} + \frac{1}{36 - (2k+1)^2} + \&c. = \frac{1}{2(2k+1)^2}.$$

182. Multiplicentur Series inventæ per nn sitque $\frac{m}{n} = p$, habebuntur istæ formæ

$$\frac{1}{1 - pp} - \frac{1}{4 - pp} + \frac{1}{9 - pp} - \frac{1}{16 - pp} + \&c. = \frac{\pi}{2p.\ fin.\ p\pi} - \frac{1}{2pp}.$$

$$\frac{1}{1 - pp} + \frac{1}{4 - pp} + \frac{1}{9 - pp} + \frac{1}{16 - pp} + \&c. = \frac{1}{2pp} - \frac{\pi}{2p.\ tang.\ p\pi}.$$

Sit $pp = a$, atque nascentur hæ Series

$$\frac{1}{1-a} - \frac{1}{4-a} + \frac{1}{9-a} - \frac{1}{16-a} + \&c. = \frac{\pi\sqrt{a}}{2a.\ fin.\ \pi\sqrt{a}} - \frac{1}{2a}$$

$$\frac{1}{1-a} + \frac{1}{4-a} + \frac{1}{9-a} + \frac{1}{16-a} + \&c. = \frac{1}{2a} - \frac{\pi\sqrt{a}}{2a.\ tang.\ \pi\sqrt{a}}.$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum Serierum per Circulum exhiberi poterit.

LIB. I.

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Cosinus Arcuum circularium supra traditam poterimus quoque summas harum Serierum assignare si a sit numerus negativus. Cum enim sit $e^{x\surd -1} = cof. x + \surd -1 \times fin. x$ & $e^{-x\surd -1} = cof. x - \surd -1 . fin. x$, erit vicissim, posito $y\surd -1$ loco x; $cof. y\surd -1 = \frac{e^{-y}+e^{y}}{2}$ & $fin. y\surd -1 = \frac{e^{-y}-e^{y}}{2\surd -1}$. Quod si ergo $a = -b$ & $y = \pi\surd b$, erit $cof. \pi\surd -b = \frac{e^{-\pi\surd b}+e^{\pi\surd b}}{2}$ & $fin. \pi\surd -b = \frac{e^{-\pi\surd b}-e^{\pi\surd b}}{2\surd -1}$; ideoque $tang. \pi\surd -b = \frac{e^{-\pi\surd b}-e^{\pi\surd b}}{(e^{-\pi\surd b}+e^{\pi\surd b})\surd -1}$. Hinc erit $\frac{\pi\surd -b}{fin. \pi\surd -b} = \frac{-2\pi\surd b}{e^{-\pi\surd b}-e^{\pi\surd b}}$; & $\frac{\pi\surd -b}{tang. \pi\surd -b} = \frac{(e^{-\pi\surd b}+e^{\pi\surd b})\pi\surd b}{e^{-\pi\surd b}-e^{\pi\surd b}}$. His ergo notatis, erit

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{\pi\surd b}{(e^{\pi\surd b}-e^{-\pi\surd b})b},$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\surd b}+e^{-\pi\surd b})\pi\surd b}{2b(e^{\pi\surd b}-e^{-\pi\surd b})} - \frac{1}{2b}.$$

Eædem hæ Series deduci possunt ex §. 162 adhibendo eandem methodum, qua in

in hoc capite ſum uſus. Quoniam vero hoc pacto reductio Sinuum & Coſinuum Arcuum imaginariorum ad quantitates exponentiales reales, non mediocriter illuſtratur, hanc explicationem alteri præferendam duxi. CAP. X.

CAPUT XI.

De aliis Arcuum atque Sinuum expreſſionibus infinitis.

184. QUONIAM ſupra (158.) denotante z Arcum Circuli quemcunque, vidimus eſſe $\mathit{ſin}.\, z = z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)$ &c., & $\mathit{coſ}.\, z = \left(1 - \frac{4zz}{\pi\pi}\right)\left(1 - \frac{4zz}{9\pi\pi}\right)\left(1 - \frac{4zz}{25\pi\pi}\right)\left(1 - \frac{4zz}{49\pi\pi}\right)$ &c., ponamus eſſe Arcum $z = \frac{m\pi}{n}$, erit $\mathit{ſin}.\, \frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{mm}{nn}\right)\left(1 - \frac{mm}{4nn}\right)\left(1 - \frac{mm}{9nn}\right)\left(1 - \frac{mm}{16nn}\right)$ &c., & $\mathit{coſ}.\, \frac{m}{n}\pi = \left(1 - \frac{4mm}{nn}\right)\left(1 - \frac{4mm}{9nn}\right)\left(1 - \frac{4mm}{25nn}\right)\left(1 - \frac{4mm}{49nn}\right)$ &c. Vel ponatur $2n$ loco n, ut prodeant hæ expreſſiones:

$$\mathit{ſin}.\, \frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{4nn - mm}{4nn}\right)\left(\frac{16nn - mm}{16nn}\right)\left(\frac{36nn - mm}{36nn}\right) \text{\&c.}$$

$$\mathit{coſ}.\, \frac{m\pi}{2n} = \left(\frac{nn - mm}{nn}\right)\left(\frac{9nn - mm}{9nn}\right)\left(\frac{25nn - mm}{25nn}\right)\left(\frac{49nn - mm}{49nn}\right) \text{\&c.},$$

quæ, in Factores ſimplices reſolutæ, dant

$$\mathit{ſin}.\, \frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{2n - m}{2n}\right)\left(\frac{2n + m}{2n}\right)\left(\frac{4n - m}{4n}\right)\left(\frac{4n + m}{4n}\right)\left(\frac{6n - m}{6n}\right) \text{\&c.}$$

LIB. I. $cof.\ \frac{m\pi}{2n} = \left(\frac{n-m}{n}\right)\left(\frac{n+m}{n}\right)\left(\frac{3n-m}{3n}\right)\left(\frac{3n+m}{3n}\right)\left(\frac{5n-m}{5n}\right)$
$\left(\frac{5n+m}{5n}\right)$ &c.

Ponatur $n-m$ loco m, quia eſt $fin.\ \frac{(n-m)\pi}{2n} = cof.\ \frac{m\pi}{2n}$ & $cof.\ \frac{(n-m)\pi}{2n} = fin.\ \frac{m\pi}{2n}$, provenient hæ expreſſiones:

$$cof.\ \frac{m\pi}{2n} = \left(\frac{(n-m)\pi}{2n}\right)\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)\left(\frac{5n-m}{4n}\right)\left(\frac{5n+m}{6n}\right) \text{ \&c.}$$

$$fin.\ \frac{m\pi}{2n} = \frac{m}{n}\left(\frac{2n-m}{n}\right)\left(\frac{2n+m}{3n}\right)\frac{4n-m}{3n}\Big)\left(\frac{4n+m}{5n}\right)\left(\frac{6n-m}{5n}\right) \text{ \&c.}$$

185. Cum igitur pro Sinu & Coſinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expreſſiones, ſi eæ inter ſe comparentur dividendo, erit $1 = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot$ &c., ideoque $\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ 12.}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ 13.}$ &c., quæ eſt expreſſio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expreſſiones exhibere licet ope primæ expreſſionis pro Sinu; ex ea enim deducitur fore:

$$\frac{\pi}{2} = \frac{n}{m} \cdot fin.\ \frac{m\pi}{2n}\left(\frac{2n}{2n-m}\right)\left(\frac{2n}{2n+m}\right)\left(\frac{4n}{4n-m}\right)\left(\frac{4n}{4n+m}\right)\left(\frac{6n}{6n-m}\right) \text{ \&c.},$$

quæ, poſito $\frac{m}{n} = 1$, præbet illam ipſam WALLISII formulam.

Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob *fin.* $\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot \text{\&c.}$$

Sit $\frac{m}{n} = \frac{1}{3}$, ob *fin.* $\frac{1}{6}\pi = \frac{1}{2}$, erit

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23} \text{ \&c.}.$$

Quod fi Expreffio *Wallifiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$; erit $\sqrt{2} = \frac{2.2.6.6.10.10.14.14.18.18.}{1.3.5.7.\ 9.11.13.15.17.19.}$ &c.

186. Quoniam Tangens cujufque Anguli æquatur Sinui per Cofinum divifo, Tangens quoque per hujufmodi Factores infinitos exprimi poterit. Quod fi autem prima Sinus expreffio dividatur per alteram Cofinus expreffionem, erit

$$tang. \frac{m\pi}{2n} = \frac{m}{n-m} \left(\frac{2n-m}{n+m}\right) \left(\frac{2n+m}{3n-m}\right) \left(\frac{4n-m}{3n+m}\right) \left(\frac{4n+m}{5n-m}\right) \text{ \&c.},$$

&

$$cot. \frac{m\pi}{2n} = \frac{n-m}{m} \left(\frac{n+m}{2n-m}\right) \left(\frac{3n-m}{2n+m}\right) \left(\frac{3n+m}{4n-m}\right) \left(\frac{5n-m}{4n+m}\right) \text{ \&c.}.$$

Simili modo autem Secantes & Cofecantes exprimentur

$$fec. \frac{m\pi}{2n} = \left(\frac{n}{n-m}\right) \left(\frac{n}{n+m}\right) \left(\frac{3n}{3n-m}\right) \left(\frac{3n}{3n+m}\right) \left(\frac{5n}{5n-m}\right) \left(\frac{5n}{5n+m}\right) \text{ \&c.}$$

$$cofec. \frac{m\pi}{2n} = \frac{n}{m} \left(\frac{n}{2n-m}\right) \left(\frac{3n}{2n+m}\right) \left(\frac{3n}{4n-m}\right) \left(\frac{5n}{4n+m}\right) \left(\frac{5n}{6n-m}\right) \text{ \&c.}$$

Sin autem alteræ Sinuum & Cofinuum formulæ combinentur, erit

$$tang.\frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdot \&c.$$

$$cot.\frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(4n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}\cdot \&c.$$

$$sec.\frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}\cdot \&c.$$

$$cosec.\frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot \&c.$$

187. Si loco m scribatur k, similique modo Anguli $\frac{k\pi}{2n}$ Sinus & Cosinus definiantur; ac per has expressiones illæ priores dividantur, prodibunt istæ formulæ

$$\frac{sin.\frac{m\pi}{2n}}{sin.\frac{k\pi}{2n}} = \frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot \&c.$$

$$\frac{sin.\frac{m\pi}{2n}}{cos.\frac{k\pi}{2n}} = \frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot \&c.$$

$$\frac{cos.\frac{m\pi}{2n}}{cos.\frac{k\pi}{2n}} = \left(\frac{n-m}{n-k}\right)\left(\frac{n+m}{n+k}\right)\left(\frac{3n-m}{3n-k}\right)\left(\frac{3n+m}{3n+k}\right)\left(\frac{5n-m}{5n-k}\right).\ \&c.$$

$$\frac{cos.\frac{m\pi}{2n}}{sin.\frac{k\pi}{2n}} = \left(\frac{n-m}{k}\right)\left(\frac{n+m}{2n-k}\right)\left(\frac{3n-m}{2n+k}\right)\left(\frac{3n+m}{4n-k}\right)\left(\frac{5n-m}{4n+k}\right).\ \&c.$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejusmodi Angulo cujus Sinus & Cosinus dentur, per hos licebit alius cujuscunque Anguli $\frac{m\pi}{2n}$ Sinum & Cosinum determinare.

188. Vicissim igitur hujusmodi expressionum, quæ ex Factoribus infinitis constant, valores veri vel per Circuli Peri-

pheriam, vel per Sinus & Cosinus Angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliæ methodi non constent, quarum ope hujusmodi productorum infinitorum valores exhiberi queant. Ceterum vero hujusmodi expressiones parum utilitatis afferunt, ad valores cum ipsius π tum Sinunm Cosinuumve Angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti Factores $\frac{\pi}{2} = 2(1-\frac{1}{9})(1-\frac{1}{25})(1-\frac{1}{49})$ &c., in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras justum invenire vellemus.

189. Præcipuus autem usus hujusmodi expressionum, etsi infinitarum, in inventione Logarithmorum versatur, in quo negotio Factorum utilitas tanta est, ut sine illis Logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit $\pi = 4(1-\frac{1}{9})(1-\frac{1}{25})(1-\frac{1}{49})$ &c., erit, sumendis Logarithmis, $l\pi = l4 + l(1-\frac{1}{9}) + l(1-\frac{1}{25}) + l(1-\frac{1}{49}) +$ &c., vel $l\pi = l2 - l(1-\frac{1}{4}) - l(1-\frac{1}{16}) - l(1-\frac{1}{36})$ — &c., sive Logarithmi communes sive hyperbolici sumantur. Quoniam vero ex Logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad Logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur, Logarithmis hyperbolicis sumendis, sit $l(1-x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ &c., si hoc modo singuli termini evolvantur, erit

$$l\pi = l4 \begin{cases} -\frac{1}{9} - \frac{1}{2 \cdot 9^2} - \frac{1}{3 \cdot 9^3} - \frac{1}{4 \cdot 9^4} - \&c. \\ -\frac{1}{25} - \frac{1}{2 \cdot 25^2} - \frac{1}{3 \cdot 25^3} - \frac{1}{4 \cdot 25^4} - \&c. \\ -\frac{1}{49} - \frac{1}{2 \cdot 49^2} - \frac{1}{3 \cdot 49^3} - \frac{1}{4 \cdot 49^4} - \&c. \\ \qquad\qquad \&c. \end{cases}$$

In his Seriebus numero infinitis verticaliter deſcendendo ejuſmodi prodeunt Series, quarum ſummas ſupra jam invenimus, quare ſi brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$
$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$
$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$
$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$
$$\&c.$$

erit $l\pi = l4 - (A - 1) - \frac{1}{2}(B - 1) - \frac{1}{3}(C - 1) - \frac{1}{4}(D - 1) - \&c.$

Eſt vero, ſummis ſupra inventis proxime exprimendis,

$A = 1,\ 2337005501361698273543\text{1}$
$B = 1,\ 0146780316041920545462\text{5}$
$C = 1,\ 0014470766409421219064\text{7}$

$A = 1,\ 23370055013616982735431$
$B = 1,\ 01467803160419205454625$
$C = 1,\ 00144707664094212190647$
$D = 1,\ 00015517902529611930298$
$E = 1,\ 00001704136304482550816$
$F = 1,\ 00000188584858311957590$
$G = 1,\ 00000020924051921150010$
$H = 1,\ 00000002323715737915670$

$I = 1,0000000025814375665977$
$K = 1,0000000002868076974558$
$L = 1,00000000003186677514044$
$M = 1,0000000000354072294392$
$N = 1,0000000000039341246691$
$O = 1,0000000000004371244859$
$P = 1,0000000000000485693682$
$Q = 1,0000000000000053965957$
$R = 1,0000000000000005996217$
$S = 1,0000000000000000666246$
$T = 1,0000000000000000074027$
$V = 1,0000000000000000008225$
$W = 1,0000000000000000000913$
$X = 1,0000000000000000000101$

Hinc fine tædiofo calculo reperitur Logarithmus hyperbolicus ipfius $\pi = 1,1447298858494001741434$, qui fi multiplicetur per $0,43429$ &c., prodit Logarithmus vulgaris ipfius $\pi = 0,497149872694133854351 26$.

191. Quia porro tam Sinum quam Cofinum Anguli $\frac{m\pi}{2n}$ expreffum habemus per Factores numero infinitos, utriufque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l\,\text{fin.}\,\frac{m\pi}{2n} = l\pi + l\frac{m}{2n} + l\left(1-\frac{mm}{4nn}\right) + l\left(1-\frac{mm}{16nn}\right) + l\left(1-\frac{mm}{36nn}\right)\ \&c.$$

$$l\,\text{cof.}\,\frac{m\pi}{2n} = l\left(1-\frac{mm}{nn}\right) + l\left(1-\frac{mm}{9nn}\right) + l\left(1-\frac{mm}{25nn}\right) + l\left(1-\frac{mm}{49nn}\right)\ \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter

LIB. I. necessitatem Series infinitas multiplicemus, terminos priores actu in Logarithmis involutos relinquamus, eritque

$$l\,fin.\frac{m\pi}{2n} = l\pi + lm + l(2n - m) + l(2n + m) - l8 - 3ln$$
$$- \frac{mm}{16nn} - \frac{m^4}{2.16^2n^4} - \frac{m^6}{3.16^3n^6} - \frac{m^8}{4.16^4n^8} - \&c.$$
$$- \frac{mm}{36nn} - \frac{m^4}{2.36^2n^4} - \frac{m^6}{3.36^3n^6} - \frac{m^8}{4.36^4n^8} - \&c.$$
$$- \frac{mm}{64nn} - \frac{m^4}{2.64^2n^4} - \frac{m^6}{3.64^3n^6} - \frac{m^8}{4.64^4n^8} - \&c.$$
$$\&c.$$

$$l\,cof.\frac{m\pi}{2n} = l(n - m) + l(n + m) - 2ln$$
$$- \frac{mm}{9nn} - \frac{m^4}{2.9^2n^4} - \frac{m^6}{3.9^3n^6} - \frac{m^8}{4.9^4n^8} - \&c.$$
$$- \frac{mm}{25nn} - \frac{m^4}{2.25^2n^4} - \frac{m^6}{3.25^3n^6} - \frac{m^8}{4.25^4n^8} - \&c.$$
$$- \frac{mm}{49nn} - \frac{m^4}{2.49^2n^4} - \frac{m^6}{3.49^3n^6} - \frac{m^8}{4.49^4n^8} - \&c.$$
$$\&c.$$

192. Occurrunt ergo in his Seriebus singulæ Potestates pares ipsius $\frac{m}{n}$, quæ sunt multiplicatæ per Series, quarum summas jam supra assignavimus. Erit nempe

$$l\,fin.\frac{m\pi}{2n} = lm + l(2n - m) + l(2n + m) - 3ln + l\pi - l8$$
$$- \frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c.\right)$$
$$- \frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c.\right)$$
$$- \frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c.\right)$$
$$- \frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c.\right)$$
$$\&c.$$

l cof.

$$l\,cos.\,\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln$$

$$-\frac{mm}{nn}\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.\right)$$
$$\&c.$$

Serierum poſteriorum modo ante (§. 190) ſummæ ſunt exhibitæ; priores Series quidem ex his derivari poſſent; at, quo facilius ad uſum tranferri queant, earum ſummas pariter hic adjiciam.

193. Quod ſi ergo, brevitatis gratia, ponamus

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \&c.$$
$$\beta = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \&c.$$
$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \&c.$$
$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \&c.$$
$$\&c.$$

erunt ſummæ in numeris proxime expreſſæ hæ:

$$\alpha = 0,41123351671205660911810$$
$$\beta = 0,06764520210694613696975$$
$$\gamma = 0,01589598534350701780804$$
$$\delta = 0,00392217717264822007570$$
$$\epsilon = 0,00097753376477325984898$$
$$\zeta = 0,00024420070472492872274$$

η	=	0, 00006103889453949332915
θ	=	0, 00001525902225127269977
ι	=	0, 00000381471182744318008
$\varkappa$	=	0, 00000095367522617534053
λ	=	0, 00000023841863595259154
μ	=	0, 00000005960464832831555
ν	=	0, 00000001490116141589813
ξ	=	0, 00000000372529031233986
o	=	0, 00000000093132257548284
π	=	0, 00000000023283064370807
ρ	=	0, 00000000005820766091685
σ	=	0, 00000000001455191522858
τ	=	0, 00000000000363797880710
υ	=	0, 00000000000090949470177
φ	=	0, 00000000000022737367544
χ	=	0, 00000000000005684341886
ψ	=	0, 00000000000001421085471
ω	=	0, 00000000000000355271367

reliquæ ſummæ in ratione quadrupla decreſcunt.

194. His ergo in ſubſidium vocatis, erit

$$l\,fin.\,\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3\,ln + l\pi - l8$$
$$- \frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(\beta - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right)$$
$$- \&c.$$

$$l\,cof.\,\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln$$
$$- \frac{mm}{nn}(A-1) - \frac{m^4}{2n^4}(B-1) - \frac{m^6}{3n^6}(C-1) - \&c.,$$

quoniam igitur Logarithmi $l\pi$ & $l8$ dantur, erit

Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n} 90^\circ =$
$lm + l(2n - m) + l(2n + m) - 3ln$

$$- \frac{m^2}{n^2} \cdot 0,93471165583043575410$$
$$- \frac{m^4}{n^4} \cdot 0,16123351671205660911$$
$$- \frac{m^6}{n^6} \cdot 0,00257260105347306848$$
$$- \frac{m^8}{n^8} \cdot 0,00009032844783567260$$
$$- \frac{m^{10}}{n^{10}} \cdot 0,00000398179316205501$$
$$- \frac{m^{12}}{n^{12}} \cdot 0,00000019425295465196$$
$$- \frac{m^{14}}{n^{14}} \cdot 0,00000001001328748812$$
$$- \frac{m^{16}}{n^{16}} \cdot 0,00000000053404135618$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,00000000002914859658$$
$$- \frac{m^{20}}{n^{20}} \cdot 0,00000000000161797979$$
$$- \frac{m^{22}}{n^{22}} \cdot 0,00000000000009097690$$
$$- \frac{m^{24}}{n^{24}} \cdot 0,00000000000000516827$$
$$- \frac{m^{26}}{n^{26}} \cdot 0,00000000000000029607$$
$$- \frac{m^{28}}{n^{28}} \cdot 0,00000000000000001708$$
$$- \frac{m^{30}}{n^{30}} \cdot 0,00000000000000000099$$
$$- \frac{m^{32}}{n^{32}} \cdot 0,00000000000000000005$$

LIB. I.

At *Logarithmus hyperbolicus Cosinus Anguli* $\frac{m}{n}\, 90^{\circ} =$

$$l(n - m) + l(n + m) - 2\,ln$$

$-\frac{m^{2}}{n^{2}}$.	0,	23370055013616982735
$-\frac{m^{4}}{n^{4}}$.	0,	00733901580209602727
$-\frac{m^{6}}{n^{6}}$.	0,	00048235888031404063
$-\frac{m^{8}}{n^{8}}$.	0,	00003879475632402982
$-\frac{m^{10}}{n^{10}}$.	0,	00000340827260896510
$-\frac{m^{12}}{n^{12}}$.	0,	00000031430809718659
$-\frac{m^{14}}{n^{14}}$.	0,	00000002989150274450
$-\frac{m^{16}}{n^{16}}$.	0,	00000000290464467239
$-\frac{m^{18}}{n^{18}}$.	0,	00000000028682639518
$-\frac{m^{20}}{n^{20}}$.	0,	00000000002868076974
$-\frac{m^{22}}{n^{22}}$.	0,	00000000000289697956
$-\frac{m^{24}}{n^{24}}$.	0,	00000000000029506024
$-\frac{m^{26}}{n^{26}}$.	0,	00000000000003026249
$-\frac{m^{28}}{n^{28}}$.	0,	00000000000000312232
$-\frac{m^{30}}{n^{30}}$.	0,	00000000000000032379
$-\frac{m^{32}}{n^{32}}$.	0,	00000000000000003373
$-\frac{m^{34}}{n^{34}}$.	0,	00000000000000000352

$$-\frac{m^{16}}{n^{16}} \cdot 0,00000000000000000037$$
$$-\frac{m^{18}}{n^{18}} \cdot 0,00000000000000000004$$

195. Si isti Sinuum & Cosinuum Logarithmi hyperbolici multiplicentur per 0, 4342944819 &c., prodibunt eorumdem Logarithmi vulgares ad Radium $= 1$ relati. Quoniam vero in Tabulis Logarithmus Sinus totius statui solet $= 10$, quo Logarithmi tabulares Sinuum & Cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis Sinus Anguli $\frac{m}{n} 90^\circ =$
$$lm + l(2n - m) + l(2n + m) - 3\,ln$$

$$+ \quad 9,594059885702190$$
$$-\frac{m^2}{n^2} \cdot 0,070022826605901$$
$$-\frac{m^4}{n^4} \cdot 0,001117266441661$$
$$-\frac{m^6}{n^6} \cdot 0,000039229146453$$
$$-\frac{m^8}{n^8} \cdot 0,000001729270798$$
$$-\frac{m^{10}}{n^{10}} \cdot 0,000000084362986$$
$$-\frac{m^{12}}{n^{12}} \cdot 0,000000004348715$$
$$-\frac{m^{14}}{n^{14}} \cdot 0,000000000231931$$
$$-\frac{m^{16}}{n^{16}} \cdot 0,000000000012659$$
$$-\frac{m^{18}}{n^{18}} \cdot 0,000000000000702$$
$$-\frac{m^{20}}{n^{20}} \cdot 0,000000000000039$$

Logarithmus tabularis Cosinus Anguli $\frac{m}{n}\, 90^\circ =$

$$l(n-m) + l(n+m) - 2\,l\,n$$

$+$		10, 000000000000000
$-$	$\frac{m^2}{n^2}.$	0, 101494859341892
$-$	$\frac{m^4}{n^4}.$	0, 003187294065451
$-$	$\frac{m^6}{n^6}.$	0, 000209485800017
$-$	$\frac{m^8}{n^8}.$	0, 000016848348597
$-$	$\frac{m^{10}}{n^{10}}.$	0, 000001480193986
$-$	$\frac{m^{12}}{n^{12}}.$	0, 000000136502272
$-$	$\frac{m^{14}}{n^{14}}.$	0, 000000012981715
$-$	$\frac{m^{16}}{n^{16}}.$	0, 000000001261471
$-$	$\frac{m^{18}}{n^{18}}.$	0, 000000000124567
$-$	$\frac{m^{20}}{n^{20}}.$	0, 000000000012456
$-$	$\frac{m^{22}}{n^{22}}.$	0, 000000000001258
$-$	$\frac{m^{24}}{n^{24}}.$	0, 000000000000128
$-$	$\frac{m^{26}}{n^{26}}.$	0, 000000000000013

196. Harum ergo formularum ope inveniri possunt Logarithmi Sinuum & Cosinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipsis Sinibus & Cosinibus. Ex Logarithmis autem Sinuum & Cosinuum per solam subtractionem inveniuntur Logarithmi Tangentium, Cotangentium, & Secantium, Cosecantiumque, quamobrem

pro his non erit opus peculiaribus formulis. Ceterum notandum est numerorum m, n, $n - m$, $n + m$, &c. Logarithmos hyperbolicos accipi oportere, cum Logarithmi hyperbolici Sinuum Cosinuumque quæruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Præterea $m : n$ denotat rationem, quam Angulus propositus habet ad Angulum rectum; sicque, cum Sinus Angulorum semirecto majorum æquentur Cosinibus Angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$, hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit. CAP. XI.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum Tangentes & Secantes quorumvis Angulorum inveniendi, quem Caput præcedens suppeditat. Quanquam enim Tangentes & Secantes per Sinus & Cosinus determinantur; tamen hoc fit per divisionem, quæ operatio in tantis numeris nimis est operosa. Ac Tangentes quidem & Cotangentes jam supra (§. 136) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic Capiti reservavimus.

198. Ex §. 181 ergo primum expressionem pro Tangente Anguli $\frac{m}{2n}\pi$ elicimus. Cum enim sit $\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\pi}{4mn}.\ tang.\ \frac{m}{2n}\pi$, erit $tang.\ \frac{m}{2n}\pi = \frac{4mn}{\pi}\left(\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c.\right)$. Cum deinde sit $\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}.\ cot.\ \frac{m}{n}\pi$, si pro n scribamus $2n$ erit $cot.\ \frac{m}{2n}\pi = \frac{2n}{m\pi} - \frac{4mn}{\pi}\left(\frac{1}{4nn - mm} + \frac{1}{16nn - mm} + \right.$

LIB. I. $\frac{1}{36nn - mm}$ + &c.). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$\text{tang.}\ \frac{m}{2n}\pi = \frac{mn}{nn-mm}\cdot\frac{4}{\pi}$$
$$+ \frac{4}{\pi}\left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \text{\&c.}\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \text{\&c.}\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \text{\&c.}\right)$$
$$\text{\&c.}$$

$$\text{cot.}\ \frac{m}{2n}\pi = \frac{n}{m}\cdot\frac{2}{\pi} - \frac{mn}{4nn-mm}\cdot\frac{4}{\pi}$$
$$- \frac{4}{\pi}\left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \text{\&c.}\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \text{\&c.}\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \text{\&c.}\right)$$
$$\text{\&c.}$$

198. At ex valore ipsius π cognito reperitur
$\frac{1}{\pi} = 0,3183098861837906715377675267450287240$,
deinde hic eædem Series occurrunt, quas supra litteris A, B, C, D, &c., & α, β, γ, δ, &c., indicavimus. His ergo notatis, erit

$\text{tang.}\ \frac{m}{2n}\pi = \frac{mn}{nn-mm}\cdot\frac{4}{\pi} + \frac{m}{n}\cdot\frac{4}{\pi}(A-1) + \frac{m^3}{n^3}\times$ $\frac{4}{\pi}(B-1) + \frac{m^5}{n^5}\cdot\frac{4}{\pi}(C-1) + \frac{m^7}{n^7}\cdot\frac{4}{\pi}(D-1)$ &c.

Deinde erit pro Cotangente

cot.

$$\text{cot.}\ \frac{m}{2n}\pi = \frac{n}{m}\cdot\frac{2}{\pi} - \frac{4mn}{4nn - mm}\cdot\frac{1}{\pi} - \frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha - \frac{1}{2^2}\right)$$
$$- \frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta - \frac{1}{2^4}\right) - \frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma - \frac{1}{2^6}\right) - \&c.,$$

atque ex his formulis natæ ſunt expreſſiones, quas ſupra (§. 135) pro Tangente & Cotangente dedimus; ſimul vero (§. 137) oſtendimus, quomodo ex Tangentibus & Cotangentibus inventis per ſolam additionem & ſubtractionem Secantes & Coſecantes reperiantur. Harum ergo regularum ope univerſus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius ſupputari poſſet, quam quidem hoc a primis conditoribus eſt factum.

CAPUT XII.

De reali Functionum fractarum evolutione.

199. JAM ſupra, in Capite ſecundo, methodus eſt tradita Functionem quamcunque fractam in tot partes reſolvendi quot ejus denominator habeat Factores ſimplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifeſtum eſt, ſi denominator quoſdam habeat Factores ſimplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo caſibus parum juvabit fractionem realem in imaginarias reſolviſſe. Cum igitur oſtendiſſem omnem Functionem integram, qualis eſt denominator cujuſvis fractionis, quantumvis Factoribus ſimplicibus imaginariis ſcateat, tamen in factores duplices, ſeu ſecundæ dimenſionis, reales ſemper reſolvi poſſe; hoc modo in reſolutione fractionum quantitates imaginariæ evitari poterunt, ſi pro denominatoribus fractionum partialium non Factores denominatoris principalis ſimplices, ſed duplices reales aſſumamus.

LIB. I.

200. Sit igitur propofita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones fimplices fecundum methodum fupra expofitam eliciantur, quot denominator N habuerit Factores fimplices reales. Sit autem, loco imaginariorum, hæc expreffio $pp - 2pqz \cdot cof. \varphi + qqzz$ Factor ipfius N; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, fit hæc fractio propofita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz \cdot cof. \varphi + qqzz)(\alpha + \beta z + \gamma zz + \delta z^3 + \&c.)},$$

ac ponatur fractio partialis ex denominatoris Factore $pp - 2pqz \cdot cof. \varphi + qqzz$ oriunda hæc: $\frac{\mathrm{A} + \mathfrak{A} z}{pp - 2pqz \cdot cof. \varphi + qqzz}$, quoniam enim variabilis z in denominatore duas habet dimenfiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam feorfim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c. = M$ & alter denominatoris Factor $\alpha + \beta z + \gamma z^2 + \&c. = \mathrm{Z}$; ponatur altera pars ex denominatoris Factore Z oriunda $= \frac{Y}{\mathrm{Z}}$, eritque $Y = \frac{M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z}{pp - 2pqz \cdot cof. \varphi + qqzz}$, quæ expreffio Functio integra ipfius z effe debet, ideoque neceffe eft ut $M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z$, divifibile fit per $pp - 2pqz \cdot cof. \varphi + qqzz$. Evanefcet ergo $M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z$, fi ponatur $pp - 2pqz \times cof. \varphi + qqzz = 0$, hoc eft fi ponatur tam $z = \frac{p}{q}(cof. \varphi + \sqrt{-1} \cdot fin. \varphi)$ quam $z = \frac{p}{q}(cof. \varphi - \sqrt{-1} \cdot fin. \varphi)$; fit $\frac{p}{q} = f$, eritque $z^n = f^n (cof. n\varphi \pm \sqrt{-1} \cdot fin. n\varphi)$. Duplex ergo hic valor pro z fubftitutus duplicem dabit æquationem, unde ambas incognitas conftantes A & $\mathfrak{A}$ definire licet.

202. Facta ergo hac substitutione, æquatio $M = AZ + \mathfrak{A}Zz$ evoluta hanc duplicem dabit æquationem

$$\left.\begin{array}{l} A + Bf.\,cos.\,\varphi + Cff.\,cos.\,2\varphi + Df^3.\,cos.\,3\varphi + \&c. \\ \pm (Bf.\,fin.\,\varphi + Cff.\,fin.\,2\varphi + Df^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \end{array}\right\} =$$

$$\left\{\begin{array}{l} A(\alpha + \beta f.\,cof.\,\varphi + \gamma ff.\,cof.\,2\varphi + \delta f^3.\,cof.\,3\varphi + \&c.) \\ \pm A\,(\beta f.\,fin.\,\varphi + \gamma ff.\,fin.\,2\varphi + \delta f^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \\ + \mathfrak{A}\,(\alpha f.\,cof.\,\varphi + \beta ff.\,cof.\,2\varphi + \gamma f^3.\,cof.\,3\varphi + \&c.) \\ \pm \mathfrak{A}\,(\alpha f.\,fin.\,\varphi + \beta ff.\,fin.\,2\varphi + \gamma f^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \end{array}\right.$$

Sit, ad calculum abbreviandum,

$$\begin{array}{r} A + Bf.\,cof.\,\varphi + Cff.\,cof.\,2\varphi + Df^3.\,cof.\,3\varphi + \&c. = P \\ Bf.\,fin.\,\varphi + Cff.\,fin.\,2\varphi + Df^3.\,fin.\,3\varphi + \&c. = \mathfrak{P} \\ \alpha + \beta f.\,cof.\,\varphi + \gamma ff.\,cof.\,2\varphi + \delta f^3.\,cof.\,3\varphi + \&c. = Q \\ \beta f.\,fin.\,\varphi + \gamma ff.\,fin.\,2\varphi + \delta f^3.\,fin.\,3\varphi + \&c. = \mathfrak{Q} \\ \alpha f.\,cof.\,\varphi + \beta ff.\,cof.\,2\varphi + \gamma f^3.\,cof.\,3\varphi + \&c. = R \\ \alpha f.\,fin.\,\varphi + \beta ff.\,fin.\,2\varphi + \gamma f^3.\,fin.\,3\varphi + \&c. = \mathfrak{R} \end{array}$$

eritque, his positis,

$$P \pm \mathfrak{P}\sqrt{-1} = AQ \pm A\mathfrak{Q}\sqrt{-1} + \mathfrak{A}R \pm \mathfrak{A}\mathfrak{R}\sqrt{-1}.$$

203. Ob signorum ambiguitatem hæ duæ oriuntur æquationes,

$$\begin{array}{l} P = AQ + \mathfrak{A}R \\ \mathfrak{P} = A\mathfrak{Q} + \mathfrak{A}\mathfrak{R} \end{array}$$

ex quibus incognitæ A & $\mathfrak{A}$ ita definiuntur, ut sit

$$A = \frac{P\mathfrak{R} - \mathfrak{P}R}{Q\mathfrak{R} - \mathfrak{Q}R} \;\&\; \mathfrak{A} = \frac{P\mathfrak{Q} - \mathfrak{P}Q}{\mathfrak{Q}R - Q\mathfrak{R}}.$$

Proposita ergo fractione $\frac{M}{(pp - 2pqz.\,cof.\,\varphi + qqzz)Z}$ per sequentem regulam fractio partialis ex ea oriunda $\frac{A + \mathfrak{A}z}{pp - 2pqz.\,cof.\,\varphi + qqzz}$ definietur. Posito $f = \frac{p}{q}$, & evo-

lutis singulis terminis, fiat ut sequitur,

$$\begin{aligned}
&\text{posito } z^n = f^n.\, cos.\, n\,\varphi, \text{ fit } & M &= \text{P}\\
&\ldots\ldots\ z^n = f^n.\, sin.\, n\,\varphi, \text{ fit } & M &= \mathrm{p}\\
&\ldots\ldots\ z^n = f^n.\, cos.\, n\,\varphi, \text{ fit } & Z &= \text{Q}\\
&\ldots\ldots\ z^n = f^n.\, sin.\, n\,\varphi, \text{ fit } & Z &= \mathrm{q}\\
&\ldots\ldots\ z^n = f^n.\, cos.\, n\,\varphi, \text{ fit } & z\,Z &= \text{R}\\
&\ldots\ldots\ z^n = f^n.\, sin.\, n\,\varphi, \text{ fit } & z\,Z &= \mathrm{r}
\end{aligned}$$

Inventis hoc modo valoribus P, Q, R, p, q, r erunt

$$\text{A} = \frac{\mathrm{P\,r} - \mathrm{p\,R}}{\mathrm{Q\,r} - \mathrm{q\,R}};\quad \mathrm{a} = \frac{\mathrm{p\,Q} - \mathrm{P\,q}}{\mathrm{Q\,r} - \mathrm{q\,R}}.$$

EXEMPLUM I.

Si fuerit proposita hæc Functio fracta $\frac{zz}{(1 - z + zz)(1 + z^4)}$ ex qua partem a denominatoris Factore $1 - z + zz$ oriundam definire oporteat, quæ sit $\frac{\text{A} + \mathrm{a}\,z}{1 - z + zz}$. Ac primo quidem hic Factor, cum forma generali $pp - 2pqz.\, cos.\,\varphi + qqzz$ comparatus, dat $p = 1$, $q = 1$ & $cos.\,\varphi = \frac{1}{2}$, unde fit $\varphi = 60^0 = \frac{\pi}{3}$. Quia itaque est $M = zz$, $Z = 1 + z^4$ & $f = 1$ erit

$$\begin{aligned}
\text{P} &= \phantom{1 +{}} cos.\tfrac{2}{3}\,\pi = -\tfrac{1}{2}; & \mathrm{p} &= \tfrac{\sqrt{3}}{2}\\
\text{Q} &= 1 + cos.\tfrac{4}{3}\,\pi = \tfrac{1}{2}; & \mathrm{q} &= -\tfrac{\sqrt{3}}{2}\\
\text{R} &= cos.\tfrac{\pi}{3} + cos.\tfrac{5\pi}{3} = 1; & \mathrm{r} &= 0.
\end{aligned}$$

Ex his invenitur $\text{A} = -1$ & $\mathrm{a} = 0$, ideoque fractio quæsita est $\frac{-1}{1 - z + zz}$, hujusque complementum erit

$\frac{1+z+zz}{1+z^4}$, cujus denominator $1+z^4$ cum habeat Factores $1+z\sqrt{2}+zz$ & $1-z\sqrt{2}+zz$, resolutio denuo suscipi potest; fit autem $\varphi=\frac{\pi}{4}$ &, priori casu $f=-1$, posteriori $f=+1$. CAP. XII.

EXEMPLUM II.

Sit igitur proposita hæc fractio resolvenda

$$\frac{1+z+zz}{(1+z\sqrt{2}+zz)(1-z\sqrt{2}+zz)}.$$

Erit $M=1+z+zz$; & pro priore Factore habebitur $f=-1$, $\varphi=\frac{\pi}{4}$ & $Z=1-z\sqrt{2}+zz$, unde erit

$$P=1-\quad cos.\frac{\pi}{4}+cos.\frac{2\pi}{4}=\frac{\sqrt{2}-1}{\sqrt{2}}$$

$$p=\quad-\quad fin.\frac{\pi}{4}+fin.\frac{2\pi}{4}=\frac{\sqrt{2}-1}{\sqrt{2}}$$

$$Q=1+\sqrt{2}.cos.\frac{\pi}{4}+cos.\frac{2\pi}{4}=2$$

$$q=\quad+\sqrt{2}.fin.\frac{\pi}{4}+fin.\frac{2\pi}{4}=2$$

$$R=-cos.\frac{\pi}{4}-\sqrt{2}.cos.\frac{2\pi}{4}-cos.\frac{3\pi}{4}=0$$

$$r=-fin.\frac{\pi}{4}-\sqrt{2}.fin.\frac{2\pi}{4}-fin.\frac{3\pi}{4}=-2\sqrt{2}$$

Ex his reperitur $Qr-qR=-4\sqrt{2}$: & $A=\frac{\sqrt{2}-1}{2\sqrt{2}}$, & $a=0$; unde ex denominatoris Factore $1+z\sqrt{2}+zz$ hæc orietur fractio partialis $\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}$, alter autem Factor dabit simili modo hanc $\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$. Hinc Functio primum proposita $\frac{zz}{(1-z+zz)(1+z^4)}$ resolvitur in has $\frac{-1}{1-z+zz}+\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}+\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$.

EXEMPLUM III.

Sit propofita hæc fractio refolvenda

$$\frac{1 + 2z + zz}{(1 - \frac{8}{5}z + zz)(1 + 2z + 3zz)}.$$

Pro Factore denominatoris $1 - \frac{8}{5}z + zz$ oriatur ifta fractio $\frac{A + \mathfrak{A}z}{1 - \frac{8}{5}z + zz}$; eritque $p = 1$; $q = 1$; $cof.\ \varphi = \frac{4}{5}$, unde $f = 1$; $M = 1 + 2z + zz$; $Z = 1 + 2z + 3zz$. Quia vero hic ratio Anguli φ ad rectum non conftat, Sinus & Cofinus ejus multiplorum feorfim debent inveftigari. Cum fit

$$cof.\ \varphi = \frac{4}{5};\ \text{erit}\ fin.\ \varphi = \frac{3}{5}$$
$$cof.\ 2\varphi = \frac{7}{25};\quad fin.\ 2\varphi = \frac{24}{25}$$
$$cof.\ 3\varphi = \frac{-44}{125};\quad fin.\ 3\varphi = \frac{117}{125};$$

hinc fit

$$P = 1 + 2.\frac{4}{5} + \frac{7}{25} = \frac{72}{25}$$
$$\mathfrak{P} = 2.\frac{3}{5} + \frac{24}{25} = \frac{54}{25}$$
$$Q = 1 + 2.\frac{4}{5} + 3.\frac{7}{25} = \frac{86}{25}$$
$$\mathfrak{Q} = 2.\frac{3}{5} + 3.\frac{24}{25} = \frac{102}{25}$$
$$R = \frac{4}{5} + 2.\frac{7}{25} - 3.\frac{44}{125} = \frac{38}{125}$$
$$\mathfrak{R} = \frac{3}{5} + 2.\frac{24}{25} + 3.\frac{117}{125} = \frac{666}{125}$$

ideoque $Q\mathfrak{R} - \mathfrak{Q}R = \frac{53400}{25.125} = \frac{2136}{125}$. Ergo

$$A = \frac{1836}{2136} = \frac{153}{178};\ \mathfrak{A} = -\frac{540}{2136} = -\frac{45}{178}.$$

Quare fractio ex Factore $1-\frac{8}{5}z+zz$ oriunda erit

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

Quaeramus simili modo fractionem alteri Factori respondentem; erit $p=1$, $q=-\sqrt{3}$ & $cof.\,\varphi=\frac{1}{\sqrt{3}}$, ergo $f=-\frac{1}{\sqrt{3}}$, $M=1+2z+zz$ & $Z=1-\frac{8}{5}z+zz$. Fiet autem, ob $cof.\,\varphi=\frac{1}{\sqrt{3}}$, $fin.\,\varphi=\frac{\sqrt{2}}{\sqrt{3}}$

$$cof.\,2\varphi=-\frac{1}{3},\ fin.\,2\varphi=\frac{2\sqrt{2}}{3}$$

$$cof.\,3\varphi=-\frac{5}{3\sqrt{3}},\ fin.\,3\varphi=\frac{\sqrt{2}}{3\sqrt{3}}$$

consequenter

$$P=1-\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}\cdot-\frac{1}{3}=\frac{2}{9}$$

$$p=\quad-\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}+\frac{1}{3}\cdot\frac{2\sqrt{2}}{3}=-\frac{4\sqrt{2}}{9}$$

$$Q=1+\frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}}+\frac{1}{3}\cdot-\frac{1}{3}=\frac{64}{45}$$

$$q=\quad+\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}+\frac{1}{3}\cdot\frac{2\sqrt{2}}{3}=\frac{24\sqrt{2}}{45}$$

$$R=-\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}-\frac{8}{5\cdot3}\cdot-\frac{1}{3}-\frac{1}{3\sqrt{3}}\cdot-\frac{5}{3\sqrt{3}}=\frac{4}{135}$$

$$r=-\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}-\frac{8}{5\cdot3}\cdot\frac{2\sqrt{2}}{3}-\frac{1}{3\sqrt{3}}\cdot\frac{\sqrt{2}}{3\sqrt{3}}=-\frac{98\sqrt{2}}{135}$$

ideoque $Qr=qR=-\frac{712\sqrt{2}}{675}$; fiet ergo

$$A=\frac{100}{712}=\frac{25}{178};\ a=\frac{540}{712}=\frac{135}{178}.$$

Fractio ergo proposita $\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$ resolvitur in $\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}+\frac{5(5+27z):178}{1+2z+3zz}$.

Lib. I. 204. Possunt autem valores litterarum R & r ex litteris Q & q definiri; cum enim sit

$$Q = \alpha + \beta f. cof. \varphi + \gamma f^2. cof. 2\varphi + \delta f^3. cof. 3\varphi \;\&c.$$
$$\mathrm{q} = \quad \beta f. fin. \varphi + \gamma f^2. fin. 2\varphi + \delta f^3. fin. 3\varphi \;\&c.$$

erit

$$Q. cof. \varphi - \mathrm{q}. fin. \varphi = \alpha. cof. \varphi + \beta f. cof. 2\varphi + \gamma f^2. cof. 3\varphi + \&c.$$

ideoque $R = f\,(Q. cof. \varphi - \mathrm{q}. fin. \varphi)$

deinde erit

$$Q. fin. \varphi + \mathrm{q}. cof. \varphi = \alpha. fin. \varphi + \beta f. fin. 2\varphi + \gamma ff. fin. 3\varphi + \&c.$$

ergo $\mathrm{r} = f.\,(Q. fin. \varphi + \mathrm{q}. cof. \varphi)$.

Ex his porro fit

$$Q\mathrm{r} - \mathrm{q}R = (QQ + \mathrm{qq})\, f. fin. \varphi$$
$$P\mathrm{r} - \mathrm{p}R = (PQ + \mathrm{pq})\, f. fin. \varphi + (P\mathrm{q} - \mathrm{p}Q)\, f. cof. \varphi$$

eritque consequenter

$$A = \frac{PQ + \mathrm{pq}}{QQ + \mathrm{qq}} + \frac{P\mathrm{q} - \mathrm{p}Q}{QQ + \mathrm{qq}} . \frac{cof. \varphi}{fin. \varphi}$$
$$\mathrm{a} = -\frac{P\mathrm{q} + \mathrm{p}Q}{(QQ + \mathrm{qq})\, f. fin. \varphi}.$$

Quare ex denominatoris Factore $pp - 2pqz. cof. \varphi + qqzz$ nascitur ista fractio partialis

$$\frac{(PQ + \mathrm{pq})\, f. fin. \varphi + (P\mathrm{q} - \mathrm{p}Q)\,(f. cof. \varphi - z)}{(pp - 2pqz. cof. \varphi + qqzz)\,(QQ + \mathrm{qq})\, f. fin. \varphi}$$

seu, ob $f = \frac{p}{q}$, hæc

$$\frac{(PQ + \mathrm{pq})\, p. fin. \varphi + (P\mathrm{q} - \mathrm{p}Q)\,(p. cof. \varphi - qz)}{(pp - 2pqz. cof. \varphi + qqzz)\,(QQ + \mathrm{qq})\, p. fin. \varphi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propositæ $\frac{M}{(pp - 2pqz. cof. \varphi + qqzz)\, Z}$ Factore denominatoris $pp - 2pqz. cof. \varphi + qqzz$, atque litteræ P, p, Q & q sequenti modo ex Functionibus M & Z inveniuntur:

posito

posito $z^n = \frac{r^n}{q^n}. cos. n\varphi$, fit $M = P$,
& $Z = Q$;

& posito $z^n = \frac{r^n}{q^n}. sin. n\varphi$, fit $M = \mathfrak{p}$,
& $Z = \mathfrak{q}$:

ubi notandum est Functiones M & Z, antequam hæc substitutio fiat, omnino evolvi debere, ut hujusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \; Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.;$$

eritque ideo

$$P = A + B\frac{p}{q}. cos.\varphi + C\frac{p^2}{q^2}. cos. 2\varphi + D\frac{p^3}{q^3}. cos. 3\varphi + \&c.$$
$$\mathfrak{p} = \quad B\frac{p}{q}. sin.\varphi + C\frac{p^2}{q^2}. sin. 2\varphi + D\frac{p^3}{q^3}. sin. 3\varphi + \&c.$$
$$Q = \alpha + \beta\frac{p}{q}. cos.\varphi + \gamma\frac{p^2}{q^2}. cos. 2\varphi + \delta\frac{p^3}{q^3}. cos. 3\varphi + \&c.$$
$$\mathfrak{q} = \quad \beta\frac{p}{q}. sin.\varphi + \gamma\frac{p^2}{q^2}. sin. 2\varphi + \delta\frac{p^3}{q^3}. sin. 3\varphi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc resolutionem locum habere non posse, si functio Z eundem Factorem $pp - 2pqz. cos.\varphi + qqzz$ adhuc in se complectatur; hoc enim casu in æquatione $M = \mathfrak{A}Z + \mathfrak{a}Zz$ facta substitutione $z^n = f^n (cos. n\varphi \pm \sqrt{-1}. sin. n\varphi)$, ipsa quantitas Z evanesceret, nihilque propterea colligi posset. Quamobrem, si Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz. cos.\varphi + qqzz)^2$ vel altiorem Potestatem, peculiari opus erit resolutione. Sit igitur $N = (pp - 2pqz. cos.\varphi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz. cos.\varphi + qqzz)^2$ oriantur hujusmodi duæ fractiones partiales

LIB. I. $\frac{A + \mathrm{a}z}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^2} + \frac{B + \mathrm{b}z}{pp - 2pqz.\,cos.\,\varphi + qqzz}$, ubi litteras constantes A, a, B, b determinari oportet.

207. His positis, debebit ista expressio

$$\frac{M - (A + \mathrm{a}z)Z - (B + \mathrm{b}z)Z(pp - 2pqz.\,cos.\,\varphi + qqzz)}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^2}$$

esse Functio integra, & hanc ob rem numerator divisibilis erit per denominatorem. Primum ergo hæc expressio $M - AZ - \mathrm{a}Zz$ divisibilis esse debet per $pp - 2pqz.\,cos.\,\varphi + qqzz$; qui cum sit casus præcedens, eodem quoque modo litteræ A & a determinabuntur.

Quare, posito $z^n = \frac{p^n}{q^n}.\,cos.\,n\varphi$, fit $M = P$,
& $Z = N$:

&, posito $z^n = \frac{p^n}{q^n}.\,fin.\,n\varphi$, fit $M = \mathrm{p}$,
& $Z = \mathrm{n}$.

Hisque factis secundum regulam supra datam, erit

$$A = \frac{P N + \mathrm{p}\,\mathrm{n}}{N + \mathrm{n}^2} + \frac{P\mathrm{n} - \mathrm{p}N}{N^2 + \mathrm{n}^2}.\frac{cos.\,\varphi}{fin.\,\varphi}$$

$$\mathrm{a} = -\frac{P\mathrm{n} + \mathrm{p}N}{N^2 + \mathrm{n}^2}.\frac{q}{p.\,fin.\,\varphi}.$$

208. Inventis ergo hoc modo A & a, fiet $\frac{M - (A + \mathrm{a}z)Z}{pp - 2pqz.\,cos.\,\varphi + qqzz}$ Functio integra, quæ sit $= P$; atque superest ut $P - BZ - \mathrm{b}Zz$ divisibile evadat per $pp - 2pqz.\,cos.\,\varphi + qqzz$, quæ expressio cum similis sit præcedenti, si

posito $z^n = \frac{p^n}{q^n}.\,cos.\,n\varphi$, vocetur $P = R$,

&, posito $z^n = \frac{p^n}{q^n}.\,fin.\,n\varphi$, vocetur $P = \mathrm{r}$; erit

$$B = \frac{\mathrm{RN} + \mathrm{rn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Rn} - \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{cos.\,\varphi}{sin.\,\varphi}$$

$$\mathrm{b} = -\frac{\mathrm{Rn} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\,sin.\,\varphi}.$$

209. Hinc jam generaliter concludere licet quomodo resolutio institui debeat, si denominator Functionis propositæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\,cos.\,\varphi + qqzz)^k$: sit enim $N = (pp - 2pqz.\,cos.\,\varphi + qqzz)^k Z$, ita ut hæc resolvenda sit Functio fracta

$$\frac{M}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^k Z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\,cos.\,\varphi + qqzz)^k$ has partes:

$$\frac{\mathrm{A} + \mathrm{a}z}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^k} + \frac{\mathrm{B} + \mathrm{b}z}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^{k-1}} + \frac{\mathrm{C} + \mathrm{c}z}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^{k-2}} + \frac{\mathrm{D} + \mathrm{d}z}{(pp - 2pqz.\,cos.\,\varphi + qqzz)^{k-3}} + \&c.$$

Jam, posito $z^n = \frac{p^n}{q^n}.\,cos.\,n\varphi$, fit $M = \mathrm{M}$,
& $z = \mathrm{N}$;

&, posito $z^n = \frac{p^n}{q^n}.\,sin.\,n\varphi$, fit $M = \mathrm{m}$,
& $z = \mathrm{n}$;

erit

$$\mathrm{A} = \frac{\mathrm{MN} + \mathrm{mn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Mn} - \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{cos.\,\varphi}{sin.\,\varphi}$$

$$\mathrm{a} = -\frac{\mathrm{Mn} + \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\,sin.\,\varphi}.$$

Deinde vocetur $\frac{M - (\mathrm{A} + \mathrm{a}z)Z}{pp - 2pqz.\,cos.\,\varphi + qqzz} = P$; atque,

posito $z^n = \frac{p^n}{q^n} . cof. n\varphi$, fit $P = \mathrm{P}$,

&, posito $z^n = \frac{p^n}{q^n} . fin. n\varphi$, fit $P = \mathrm{p}$;

erit

$$\mathrm{B} = \frac{\mathrm{PN} + \mathrm{pN}}{\mathrm{N}^2 + \mathrm{N}^2} + \frac{\mathrm{PN} - \mathrm{pN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{cof. \varphi}{fin. \varphi}$$

$$\mathrm{b} = - \frac{\mathrm{PN} + \mathrm{pN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{q}{p . fin. \varphi}.$$

Tum vocetur $\frac{P - (\mathrm{B} + \mathrm{b}z) Z}{pp - 2pqz . cof. \varphi + qqzz} = Q$, atque

posito $z^n = \frac{p^n}{q^n} . cof. n\varphi$, fit $Q = \mathrm{Q}$,

&, posito $z^n = \frac{p^n}{q^n} . fin. n\varphi$, fit $Q = \mathrm{q}$;

erit

$$\mathrm{C} = \frac{\mathrm{QN} + \mathrm{qN}}{\mathrm{N}^2 + \mathrm{N}^2} + \frac{\mathrm{QN} - \mathrm{qN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{cof. \varphi}{fin. \varphi}$$

$$\mathrm{c} = - \frac{\mathrm{QN} + \mathrm{qN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{q}{p . fin. \varphi}.$$

Porro vocetur $\frac{Q - (\mathrm{C} + \mathrm{c}z) Z}{pp - 2pqz . cof. \varphi + qqzz} = R$, atque

posito $z^n = \frac{p^n}{q^n} . cof. n\varphi$, fit $R = \mathrm{R}$,

&, posito $z^n = \frac{p^n}{q^n} . fin. n\varphi$, fit $R = \mathrm{r}$;

erit

$$\mathrm{D} = \frac{\mathrm{RN} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{N}^2} + \frac{\mathrm{RN} - \mathrm{rN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{cof. \varphi}{fin. \varphi}$$

$$\mathrm{d} = - \frac{\mathrm{RN} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{N}^2} . \frac{q}{p . fin. \varphi}.$$

Hocque modo progrediendum est donec ultimæ fractionis, cujus denominator est $pp - 2pqz. \cos.\varphi + qqzz$, numerator fuerit determinatus.

EXEMPLUM.

Sit ista proposita Functio fracta

$$\frac{z - z^3}{(1 + zz)^4 (1 + z^4)}$$

ex cujus denominatoris Factore $(1 + zz)^4$ oriantur hæ fractiones partiales,

$$\frac{\mathrm{A} + \mathfrak{A}z}{(1 + zz)^4} + \frac{\mathrm{B} + \mathfrak{B}z}{(1 + zz)^3} + \frac{\mathrm{C} + \mathfrak{C}z}{(1 + zz)^2} + \frac{\mathrm{D} + \mathfrak{D}z}{1 + zz}.$$

Comparatione ergo instituta, erit $p = 1$, $q = 1$, $\cos.\varphi = 0$; ideoque $\varphi = \frac{1}{2}\pi$, porroque $M = z - z^3$ & $\mathrm{Z} = 1 + z^4$. Hinc erit $\mathrm{M} = 0$; $\mathfrak{M} = 2$; $\mathrm{N} = 2$; $\mathfrak{N} = 0$, & $\sin.\varphi = 1$.

Hinc itaque invenitur

$$\mathrm{A} = -\frac{4}{4}.\, 0 = 0, \ \& \ \mathfrak{A} = 1.$$

Ergo $\mathrm{A} + \mathfrak{A}z = z$; hincque $P = \frac{z - z^3 - z - z^3}{1 + zz} = -z^3$, & $\mathrm{P} = 0$, $\mathfrak{P} = 1$: unde reperitur

$$\mathrm{B} = 0, \ \& \ \mathfrak{B} = \frac{1}{2}.$$

Ergo $\mathrm{B} + \mathfrak{B}z = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z - \frac{1}{2}z^3$;

unde $\mathrm{Q} = 0$ & $\mathfrak{Q} = 0$, ergo $\mathrm{C} = 0$ & $\mathfrak{C} = 0$, hincque $R = -\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z$,

ergo $\mathrm{R} = 0$; $\mathfrak{R} = -\frac{1}{2}$; unde fit

$$\mathrm{D} = 0 \ \& \ \mathfrak{D} = -\frac{1}{4}.$$

LIB. I. Quamobrem fractiones quæsitæ sunt hæ
$\frac{z}{(1+zz)^3} + \frac{z}{2(1+zz)^2} - \frac{z}{4(1+zz)}$. Reliquæ vero fractionis numerator est $= S = \frac{R - (\mathfrak{D} + \mathfrak{d}z)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$, quæ ergo erit $= \frac{-z+z^3}{4(1+z^4)}$.

210. Hac ergo methodo simul innotescit fractio complementi, quæ cum inventis conjuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp - 2pqz.\, cos.\, \varphi + qqzz)^k Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz.\, cos.\, \varphi + qqzz)^k$ oriundæ, pro quibus formati sunt valores Functionum P, Q, R, S, T, si harum litterarum Series ulterius continuetur, erit ea, quæ ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, si $k = 1$, erit reliqua fractio $\frac{P}{Z}$; si $k = 2$, erit reliqua fractio $\frac{Q}{Z}$; si $k = 3$, erit ea $\frac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius resolvi poterit.

CAPUT XIII.

De Seriebus recurrentibus.

211. Ad hoc Seriemum genus, quas MOIVRÆUS *recurrentes* vocare solet, hic refero omnes Series quæ ex evolutione Functionis cujusque fractæ per divisionem actualem instituta nascuntur. Supra enim jam ostendimus has Series ita esse comparatas, ut quivis terminus ex aliquot præcedentibus secundum legem quandam constantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias simpliciores resolvere docuerim, hinc Series quoque recurrens in alias simpliciores resolvetur. In hoc igitur Capite propositum est Serierum recurrentium cujusvis gradus resolutionem in simpliciores exponere.

212. Sit proposita ista Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per divisionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, supra est ostensum. Quod si jam Functio illa fracta resolvatur in fractiones suas simplices, & unaquæque in Seriem recurrentem evolvatur, manifestum est summam omnium harum Serierum ex fractionibus partialibus ortarum æqualem esse debere Seriei recurrenti

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas supra invenire docuimus, dabunt

Series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem Series partiales junctim sumtæ producent Seriem recurrentem propositam; unde & hujus natura penitius cognoscetur.

213. Sint Series recurrentes ex singulis fractionibus partialibus ortæ hæ.

$$\begin{array}{l} \mathrm{a} + \mathrm{b}z + \mathrm{c}zz + \mathrm{d}z^3 + \mathrm{e}z^4 + \&c. \\ \mathrm{a}' + \mathrm{b}'z + \mathrm{c}'zz + \mathrm{d}'z^3 + \mathrm{e}'z^4 + \&c. \\ \mathrm{a}'' + \mathrm{b}''z + \mathrm{c}''zz + \mathrm{d}''z^3 + \mathrm{e}''z^4 + \&c. \\ \mathrm{a}''' + \mathrm{b}'''z + \mathrm{c}'''zz + \mathrm{d}'''z^3 + \mathrm{e}'''z^4 + \&c. \\ \&c. \end{array}$$

Quoniam hæ Series junctim sumtæ æquales esse debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \&c.$$

necesse est ut sit

$$\begin{array}{l} A = \mathrm{a} + \mathrm{a}' + \mathrm{a}'' + \mathrm{a}''' + \&c. \\ B = \mathrm{b} + \mathrm{b}' + \mathrm{b}'' + \mathrm{b}''' + \&c. \\ C = \mathrm{c} + \mathrm{c}' + \mathrm{c}'' + \mathrm{c}''' + \&c. \\ D = \mathrm{d} + \mathrm{d}' + \mathrm{d}'' + \mathrm{d}''' + \&c. \\ \&c. \end{array}$$

Hinc, si singularum Serierum ex fractionibus partialibus ortarum definiri queant coëfficientes Potestatis z^n, horum summa dabit coëfficientem Potestatis z^n in Serie recurrente $A + Bz + Cz^2 + Dz^3 + \&c.$

214. Dubium hic suboriri posset, an, si duæ hujusmodi Series fuerint inter se æquales

$$A + Bz + Cz^2 + Dz^3 + \&c. = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \&c.,$$

necessario inde sequatur, coëfficientes similium Potestatum ipsius z inter se esse æquales; seu an sit $A = \mathrm{A}$; $B = \mathrm{B}$; $C = \mathrm{C}$;

$C = C$; $D = D$; &c.. Hoc autem dubium facile tolletur, si perpendamus hanc æqualitatem subsistere debere quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$, atque manifestum est fore $\mathcal{A} = A$. His ergo terminis æqualibus utrinque sublatis, ac reliqua æquatione per z divisa, habebitur $\mathcal{B} + \mathcal{C}z + \mathcal{D}z^2 + \&c. = B + Cz + Dz^2 + \&c.$; unde sequitur fore $\mathcal{B} = B$: simili autem modo ostendetur esse $\mathcal{C} = C$; $\mathcal{D} = D$, & ita porro in infinitum.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem $\frac{A}{1 - pz}$ dare Seriem $A + Apz + Ap^2z^2 + Ap^3z^3 + \&c.$, cujus terminus generalis est Ap^nz^n; hæc enim expressio vocari solet *terminus generalis*, quoniam ex ea, loco n numeros omnes successive substituendo, omnes Seriei termini nascuntur. Deinde ex fractione $\frac{A}{(1 - pz)^2}$ oritur Series $A + 2Apz + 3Ap^2z^2 + 4Ap^3z^3 + \&c.$, cujus terminus generalis est $(n + 1)Ap^nz^n$. Tum ex fractione $\frac{A}{(1 - pz)^3}$ oritur Series $A + 3Apz + 6Ap^2z^2 + 10Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n + 1)(n + 2)}{1 \cdot 2} Ap^nz^n$. Generatim autem fractio $\frac{A}{(1 - pz)^k}$ præbet Seriem hanc $A + kApz + \frac{k(k + 1)}{1 \cdot 2} Ap^2z^2 + \frac{k(k + 1)(k + 2)}{1 \cdot 2 \cdot 3} Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n + 1)(n + 2)(n + 3) \ldots \ldots (n + k - 1)}{1 \cdot 2 \cdot 3 \ldots \ldots (k - 1)} Ap^nz^n$. Ex ipsa autem Seriei progressione colligitur hic idem terminus $= \frac{k(k + 1)(k + 2) \ldots \ldots (k + n - 1)}{1 \cdot 2 \cdot 3 \ldots \ldots n} Ap^nz^n$: hæc vero

LIB. I. expressio illi est æqualis, id quod eliminatis denominatoribus patebit, fiet enim,

$$1.2.3.....n(n+1).....(n+k-1) = 1.2.3.....(k-1)k.....(k+n-1)$$

quæ est æquatio identica.

216. Quoties ergo in resolutione Functionum fractarum ad hujusmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 +$ &c., terminus generalis assignari poterit, quippe qui erit summa terminorum generalium Serierum, quæ ex fractionibus partialibus nascuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ *nascitur.*

Series hinc nata est $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 +$ &c.. Ad coëfficientem potestatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ resolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæsitus $(\frac{2}{3}(-1)^n + \frac{1}{3}.2^n)z^n = \frac{2^n \pm 2}{3}z^n$, ubi signum + valet si n sit numerus par, signum − si n sit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, *seu Seriei hujus* $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 +$ &c.

Ob denominatorem $= (1 - 2z)(1 - 3z)$ resolvitur fractio in has $\frac{-1}{1-2z} + \frac{2}{1-3z}$, ex quibus fit terminus generalis $2.\,3^n z^n - 2^n z^n = (2.\,3^n - 2^n) z^n$. CAP. XIII.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - \left(\frac{1+\sqrt{5}}{2}\right) z$ & $1 - \left(\frac{1-\sqrt{5}}{2}\right) z$, per resolutionem prodeunt $\frac{\frac{1+\sqrt{5}}{2}}{1-\left(\frac{1+\sqrt{5}}{2}\right)z} + \frac{\frac{1-\sqrt{5}}{2}}{1-\left(\frac{1-\sqrt{5}}{2}\right)z}$, unde erit terminus generalis $=$
$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} z^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} z^n$.

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus
$a + (\alpha a + b)z + (\alpha^2 a + \alpha b + \zeta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\zeta a + \zeta b)z^3 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{a+bz}{1-\alpha z-\zeta zz}$.

Per resolutionem oriuntur hæ duæ fractiones:

 $\frac{(a(\alpha+\sqrt{(\alpha\alpha+4\beta)})+2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})z}+$

$\frac{(a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})z}$, hinc terminus generalis erit $\frac{a(\sqrt{(\alpha\alpha+4\beta)}+\alpha)+2b}{2\sqrt{(\alpha\alpha+4\beta)}}\times$ $(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n+\frac{a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b}{2\sqrt{(\alpha\alpha+4\beta)}}\times$ $(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n$; ex quo omnium Serierum recurrentium, quarum quisque terminus per duos præcedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM V.

Invenire terminum generalem hujus Seriei $1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+$ &c., *quæ oritur ex fractione* $\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}$.

Quanquam lex progressionis primo intuitu ita est manifesta ut explicatione non indigeat, tamen fractiones per resolutionem ortæ $\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$ dant hunc terminum generalem $\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n=$ $\frac{2n+3\pm1}{4}z^n$, ubi signum superius valet si n fuerit numerus par, inferius si n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in hujusmodi fractiones partiales simplices resolvere licet. Quod si autem expressiones imaginarias vitare velimus, sæpenumero ad hujusmodi fractiones partiales pervenietur

$\frac{A + Bpz}{1 - 2pz.\,cof.\,\varphi + ppzz}$; $\frac{A + Bpz}{(1 - 2pz.\,cof.\,\varphi + ppzz)^2}$; &c. $\frac{A + Bpz}{(1 - 2pz.\,cof.\,\varphi + ppzz)^k}$, ex quarum evolutione cujusmodi Series nascantur videndum est. Ac primo quidem, ob $cof.\,n\varphi = 2\,cof.\,\varphi.\,cof.\,(n-1)\varphi - cof.\,(n-2)\varphi$, fractio $\frac{A}{1 - 2pz.\,cof.\,\varphi + ppzz}$ evoluta dabit

$$\begin{array}{l} A + 2Apz.\,cof.\,\varphi + 2Appzz.\,cof.\,2\varphi + 2Ap^3z^3.\,cof.\,3\varphi + 2Ap^4z^4.\,cof.\,4\varphi \\ \qquad\quad + Appzz \qquad\qquad + 2Ap^3z^3.\,cof.\,\varphi + 2Ap^4z^4.\,cof.\,2\varphi \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + Ap^4z^4 \qquad \&c. \end{array}$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas Series

$$Ppz.\,fin.\,\varphi + Pp^2z^2.\,fin.\,2\varphi + Pp^3z^3.\,fin.\,3\varphi + Pp^4z^4.\,fin.\,4\varphi + \&c.$$
$$Q + Qpz.\,cof.\,\varphi + Qp^2z^2.\,cof.\,2\varphi + Qp^3z^3.\,cof.\,3\varphi + Qp^4z^4.\,cof.\,4\varphi + \&c.$$

quæ duæ Series utique nascuntur ex evolutione fractionis, cujus denominator est $1 - 2pz.\,cof.\,\varphi + ppzz$. Ac prior quidem oritur ex hac fractione $\frac{Ppz.\,fin.\,\varphi}{1 - 2pz.\,cof.\,\varphi + ppzz}$, posterior vero ex hac $\frac{Q - Qpz.\,cof.\,\varphi}{1 - 2pz.\,cof.\,\varphi + ppzz}$. Addantur hæ duæ fractiones, atque summa $\frac{Q + Ppz.\,fin.\,\varphi - Qpz.\,cof.\,\varphi}{1 - 2pz.\,cof.\,\varphi + ppzz}$ dabit Seriem cujus terminus generalis erit $= (P\,fin.\,n\varphi + Q\,cof.\,n\varphi)\,p^n z^n$. Fiat autem hæc fractio propositæ $\frac{A + Bpz}{1 - 2pz.\,cof.\,\varphi + ppzz}$ æqualis, erit $Q = A$, & $P = A\,cot.\,\varphi + B\,cofec.\,\varphi$. Seriei ergo ex hac fractione $\frac{A + Bpz}{1 - 2pz.\,cof.\,\varphi + ppzz}$ ortæ terminus generalis erit $= \frac{A\,cof.\,\varphi.\,fin.\,n\varphi + B\,fin.\,n\varphi + A\,fin.\,\varphi.\,cof.\,n\varphi}{fin.\,\varphi}\,p^n z^n = \frac{A\,fin.\,(n+1)\varphi + B\,fin.\,n\varphi}{fin.\,\varphi}\,p^n z^n$.

LIB. I. 219. Ad terminum generalem inveniendum, si denominator fractionis fuerit Potestas, ut $(1 - 2pz.\,cof.\,\varphi + ppzz)^k$, conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{(1-(cof.\,\varphi+\sqrt{-1}.\,fin.\,\varphi)pz)^k} + \frac{b}{(1-(cof.\,\varphi-\sqrt{-1}.\,fin.\,\varphi)pz)^k},$$

quarum simul sumtarum terminus generalis Seriei ex ipsis ortæ erit

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(cof.\,n\varphi+\sqrt{-1}.\,fin.\,n\varphi)ap^nz^n +$$
$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(cof.\,n\varphi-\sqrt{-1}.\,fin.\,n\varphi)bp^nz^n.$$

Sit $a + b = f$; $a - b = \frac{g}{\sqrt{-1}}$, ut sit $a = \frac{f\sqrt{-1}+g}{2\sqrt{-1}}$ & $b = \frac{f\sqrt{-1}-g}{2\sqrt{-1}}$, eritque hæc expressio

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(f.\,cof.\,n\varphi+g.\,fin.\,n\varphi)p^nz^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{(1-(cof.\,\varphi+\sqrt{-1}.\,fin.\,\varphi)pz)^k} + \frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{(1-(cof.\,\varphi-\sqrt{-1}.\,fin.\,\varphi)pz)^k},$$

seu quæ oritur ex hac fractione una

$$\frac{\begin{array}{l}f-kfpz.cof.\varphi+\frac{k(k-1)}{1.\,2}fp^2z^2.cof.2\varphi-\frac{k(k-1)(k-2)}{1.\,2.\,3}fp^3z^3.cof.3\varphi\\ +kgpz.fin.\varphi-\frac{k(k-1)}{1.\,2}gp^2z^2.fin.2\varphi+\frac{k(k-1)(k-2)}{1.\,2.\,3}gp^3z^3.fin.3\varphi\ \&c,\end{array}}{(1-2pz.\,cof.\,\varphi+ppzz)^k}$$

220. Posito ergo $k = 2$, erit Seriei ex hac fractione

$$\frac{f - 2pz(f.\,cof.\,\varphi - g.\,fin.\,\varphi) + ppzz(f.\,cof.\,2\varphi - g.\,fin.\,2\varphi)}{(1 - 2pz.\,cof.\,\varphi + ppzz)^2}$$

ortæ terminus generalis $= (n+1)(f.cof.n\varphi + g.fin.n\varphi)p^nz^n$.
At Seriei ex hac fractione $\frac{a}{1-2pz.\,cof.\,\varphi+ppzz}$, seu hac

$\frac{a - 2apz.\cos.\varphi + appzz}{(1 - 2pz.\cos.\varphi + ppzz)^2}$ ortæ terminus generalis eſt $=$ CAP. XIII.
$\frac{a.\sin.(n+1)\varphi}{\sin.\varphi} p^n z^n$. Addantur hæ fractiones invicem, ac ponatur $a + f = A$; $2a.\cos.\varphi + 2f.\cos.\varphi - 2g.\sin.\varphi = -B$ & $a + f.\cos.2\varphi - g.\sin.2\varphi = 0$, hinc erit $g =$ $\frac{B + 2A\cos.\varphi}{2\sin.\varphi}$, $a = \frac{A + B\cos.\varphi}{1 - \cos.2\varphi} = \frac{A + B\cos.\varphi}{2(\sin.\varphi)^2}$ & $f = -\frac{A\cos.2\varphi - B\cos.\varphi}{2(\sin.\varphi)^2}$, & $g = \frac{B\sin.\varphi + A\sin.2\varphi}{2(\sin.\varphi)^2}$. Hanc ob rem Seriei ex hac fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\varphi + ppzz)^2}$, ortæ terminus generalis eſt $\frac{A + B\cos.\varphi}{2(\sin.\varphi)^2}\sin.(n+1)\varphi.p^n z^n + (n+1) \times$
$\frac{(B\sin.\varphi.\sin.n\varphi + A\sin.2\varphi.\sin.n\varphi - B\cos.\varphi.\cos.n\varphi - A\cos.2\varphi.\cos.n\varphi)}{2(\sin.\varphi)^2} \times$
$p^n z^n = -\frac{(n+1)(A\cos.(n+2)\varphi + B\cos.(n+1)\varphi)}{2(\sin.\varphi)^2} p^n z^n +$
$\frac{(A + B\cos.\varphi)\sin.(n+1)\varphi}{2(\sin.\varphi)^3} p^n z^n =$
$\frac{(\frac{1}{2}(n+3)\sin.(n+1)\varphi - \frac{1}{2}(n+1)\sin.(n+3)\varphi)}{2(\sin.\varphi)^3} A p^n z^n +$
$\frac{(\frac{1}{2}(n+2)\sin.n\varphi - \frac{1}{2}n\sin.(n+2)\varphi)}{2(\sin.\varphi)^3} B p^n z^n$. Eſt ergo iſte terminus generalis quæſitus $=$
$\frac{(n+3)\sin.(n+1)\varphi - (n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3} A p^n z^n +$
$\frac{(n+2)\sin.n\varphi - n.\sin.(n+2)\varphi}{4(\sin.\varphi)^3} B p^n z^n$ Seriei quæ oritur ex fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\varphi + ppzz)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ
$\frac{f - 3pz(f.\cos.\varphi - g.\sin.\varphi) + 3ppzz(f.\cos.2\varphi - g.\sin.2\varphi) - p^3z^3(f.\cos.3\varphi - g.\sin.3\varphi)}{(1 - 2pz.\cos.\varphi + ppzz)^3}$
terminus generalis $= \frac{(n+1)(n+2)}{1.\ \ 2}(f.\cos.n\varphi + g.\sin.n\varphi)p^n z^n$.

LIB. I. Deinde Seriei ex fractione $\frac{a+bpz}{(1-2pz.\,cos.\varphi+ppzz)^3}$, seu ex hac

$$\frac{a-2apz.\,cos.\varphi+appzz+bpz-2bppzz.\,cos.\varphi+bp^3z^3}{(1-2pz.\,cos.\varphi+ppzz)^3}$$

ortæ terminus generalis eſt $\frac{(n+3)\,sin.(n+1)\varphi-(n+1)\,sin.(n+3)\varphi}{4(sin.\varphi)^3}ap^nz^n+\frac{(n+2)\,sin.n\varphi-n.\,sin.(n+2)\varphi}{4(sin.\varphi)}bp^nz^n$. Addantur hæ fractiones ac ponatur numerator $=$ A, erit $a+f=$ A, $3f.cos.\varphi-3g.sin.\varphi+2a.cos.\varphi-b=0$, $3f.cos.2\varphi-3g.sin.2\varphi+a-2b.\,cos.\varphi=0$ & $b=f.cos.3\varphi-g.sin.3\varphi$, hinc erit $a=\frac{f\,cos.3\varphi-g.\,sin.3\varphi-3f.\,cos.\varphi+3g.\,sin.\varphi}{2\,cos.\varphi}=2g.(sin.\varphi)^3\,tang.\varphi-f-2f.(sin.\varphi)^2$. Deinde reperitur $\frac{f}{g}=\frac{sin.5\varphi-2\,sin.3\varphi+sin.\varphi}{cos.5\varphi-2\,cos.3\varphi+cos.\varphi}$ & $a+f=$ A $=2g.(sin.\varphi)^3\,tang.\varphi-2f(sin.\varphi)^2$; ergo $\frac{A}{2(sin.\varphi)^2}=\frac{g.\,sin.\varphi-f.\,cos.\varphi}{cos.\varphi}$; ex quibus tandem oritur $f=\frac{A(sin.\varphi-2\,sin.3\varphi+sin.5\varphi)}{16(sin.\varphi)^5}$, $g=\frac{A(cos.\varphi-2\,cos.3\varphi+cos.5\varphi)}{16(sin.\varphi)^5}$, ob $16(sin.\varphi)^5=sin.5\varphi-5\,sin.3\varphi+10\,sin.\varphi$, erit $a=\frac{A(9\,sin.\varphi-3\,sin.3\varphi)}{10(sin.\varphi)^5}$ & $b=\frac{A(-sin.2\varphi+sin.2\varphi)}{16(sin.\varphi)^5}=0$. Eſt autem $3\,sin.\varphi-sin.3\varphi=4(sin.\varphi)^3$; ergo $a=\frac{3A}{4(sin.\varphi)^2}$. Quocirca erit terminus generalis $\frac{(n+1)(n+2)}{1.\quad 2}p^nz^n\times$

$$A\frac{(sin.(n+1)\varphi-2\,sin.(n+3)\varphi+sin.(n+5)\varphi)}{16(sin.\varphi)^5}+$$

$$3Ap^nz^n.\frac{((n+3)\,sin.(n+1)\varphi-(n+1)\,sin.(n+3)\varphi)}{16(sin.\varphi)^5}=$$

$$\frac{Ap^nz^n}{16(sin.\varphi)^5}\left(\frac{(n+4)(n+5)}{1.\quad 2}sin.(n+1)\varphi-\frac{2(n+1)(n+5)}{1.\quad 2}\times sin.(n+3)\varphi+\frac{(n+1)(n+2)}{1.\quad 2}sin.(n+5)\varphi\right).$$

222. Seriei

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\,cof.\,\varphi + ppzz)^3}$$

terminus generalis erit hic

$$\frac{Ap^n z^n}{16\,(fin.\,\varphi)^5}\left(\frac{(n+5)(n+4)}{1.\;\;2} fin.\,(n+1)\varphi - \frac{2(n+1)(n+5)}{1.\;\;2} \times \right.$$
$$fin.\,(n+3)\varphi + \frac{(n+1)(n+2)}{1.\;\;2} fin.\,(n+5)\varphi\Big)$$
$$+ \frac{Bp^n z^n}{16\,(fin.\,\varphi)^5}\Big(\frac{(n+4)(n+3)}{1.\;\;2} fin.\,n\varphi - \frac{2n(n+4)}{1.\;\;2} fin.\,(n+2)\varphi +$$
$$\left.\frac{n(n+1)}{1.\;\;2} fin.\,(n+4)\varphi\right).$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\,cof.\,\varphi + ppzz)^4}$$

terminus generalis erit hic

$$+ \frac{Ap^n z^n}{64\,(fin.\,\varphi)^7}\Big(\frac{(n+7)(n+6)(n+5)}{1.\;\;2.\;\;3} fin.\,(n+1)\varphi -$$
$$\frac{3(n+1)(n+7)(n+6)}{1.\;\;2.\;\;3} fin.\,(n+3)\varphi + \frac{3(n+1)(n+2)(n+7)}{1.\;\;2.\;\;3} \times$$
$$fin.\,(n+5)\varphi - \frac{(n+1)(n+2)(n+3)}{1.\;\;2.\;\;3} fin.\,(n+7)\varphi\Big)$$
$$+ \frac{Bp^n z^n}{64\,(fin.\,\varphi)^7}\Big(\frac{(n+6)(n+5)(n+4)}{1.\;\;2.\;\;3} fin.\,n\varphi -$$
$$\frac{3n(n+6)(n+5)}{1.\;\;2.\;\;3} fin.\,(n+2)\varphi + \frac{3n(n+1)(n+6)}{1.\;\;2.\;\;3} \times$$
$$fin.\,(n+4)\varphi - \frac{n(n+1)(n+2)}{1.\;\;2.\;\;3} fin.\,(n+6)\varphi\Big).$$

Ex his autem expreſſionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expreſſionum penitius inſpiciendam notari convenit eſſe

LIB. I.

$$\begin{aligned}
\sin.\varphi &= \sin.\varphi \\
4(\sin.\varphi)^3 &= 3\sin.\varphi - \sin.3\varphi \\
16(\sin.\varphi)^5 &= 10\sin.\varphi - 5\sin.3\varphi + \sin.5\varphi \\
64(\sin.\varphi)^7 &= 35\sin.\varphi - 21\sin.3\varphi + 7\sin.5\varphi - \sin.7\varphi \\
256(\sin.\varphi)^9 &= 126\sin.\varphi - 84\sin.3\varphi + 36\sin.5\varphi - 9\sin.7\varphi + \sin.9\varphi \\
&\&c.
\end{aligned}$$

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales resolvi queant, simul omnium Serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adjuncta sunt.

EXEMPLUM I.

Ex fractione $\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$, oritur ista Series recurrens $1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+$ &c.; cujus terminus generalis desideratur. Fractio proposita secundum Factores ordinata fit $= \frac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ resolvitur in has fractiones $\frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{17}{72(1-z)} + \frac{1}{8(1+z)} + \frac{(2+z)}{9(1+z+zz)}$. Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem $\frac{(n+1)(n+2)}{1.\quad 2}\cdot\frac{1}{6}z^n = \frac{nn+3n+2}{12}z^n$: secunda $\frac{1}{4(1-z)^2}$ dat $\frac{n+1}{4}z^n$: tertia $\frac{17}{72(1-z)}$ dat $\frac{17}{72}z^n$: quarta $\frac{1}{8(1+z)}$ dat $\frac{1}{8}(-1)^n z^n$.

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma $\frac{A+Bpz}{1-2pz.\,cof.\,\varphi+ppzz}$ (218) dat $p=1$, $\varphi=\frac{\pi}{3}=60^{\circ}$; $A=+\frac{2}{9}$ & $B=-\frac{1}{9}$, unde oritur terminus generalis $+\frac{2\,fin.\,(n+1)\varphi-fin.\,n\varphi}{9\,fin.\,\varphi}(-1)^n z^n=\frac{4\,fin.\,(n+1)\varphi-2\,fin.\,n\varphi}{9}\times$ $(-1)^n z^n=\frac{4\,fin.\,(n+1)\frac{\pi}{3}-2\,fin.\,n\frac{\pi}{3}}{9}(-1)^n z^n$. Colligantur hæ expreſſiones omnes in unam ſummam, ac prodibit Seriei propoſitæ terminus generalis quæſitus $=(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72})z^n\pm\frac{1}{8}z^n\pm\frac{4\,fin.\,(n+1)\frac{\pi}{3}-2\,fin.\,n\frac{\pi}{3}}{9}z^n$, ubi ſigna ſuperiora valent ſi n numerus par, inferiora ſi n impar. Ubi notandum eſt ſi fuerit n numerus formæ $3m$ fore $\frac{4\,fin.\,\frac{1}{3}(n+1)\pi-2\,fin.\,\frac{1}{3}n\pi}{9}=\pm\frac{2}{9}$; ſi fuerit $n=3m+1$ ſeu $n=3m+2$, erit iſta expreſſio $=\mp\frac{1}{9}$, prout n fuerit numerus vel par vel impar. Ex his natura Seriei ita explicari poteſt, ut

ſi fuerit	terminus generalis futurus ſit
$n = 6\,m + 0$	$(\frac{nn}{12} + \frac{n}{2} + 1)\, z^n$
$n = 6\,m + 1$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12})\, z^n$
$n = 6\,m + 2$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3})\, z^n$
$n = 6\,m + 3$	$(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4})\, z^n$
$n = 6\,m + 4$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3})\, z^n$
$n = 6\,m + 5$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12})\, z^n$.

Sic, ſi fuerit $n = 50$, valet forma $n = 6\,m + 2$, eritque terminus Seriei $= 234\, z^{50}$.

Exemplum II.

Ex fractione $\frac{1 + z + zz}{1 - z - z^3 + z^4}$ oritur hæc Series recurrens $1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$, cujus terminum generalem invenire oportet. Fractio propoſita ad hanc formam reducitur $\frac{1 + z + zz}{(1 - z)^2\,(1 + z)\,(1 + zz)}$, quæ propterea reſolvitur in has fractiones partiales $\frac{3}{4(1 - z)^2} + \frac{3}{8(1 - z)} + \frac{1}{8(1 + z)} - \frac{1 + z}{4(1 + zz)}$. Harum prima $\frac{3}{4(1 - z)^2}$ dat terminum generalem $\frac{3(n+1)}{4} z^n$; ſecunda $\frac{3}{8(1 - z)}$ dat $\frac{3}{8} z^n$; tertia dat $\frac{1}{8}(-1)^n z^n$; & quarta $-\frac{1 + z}{4(1 + zz)}$ comparata cum forma $\frac{A + Bpz}{1 - 2pz\, cof.\, \varphi + ppzz}$ dat $p = 1$, $cof.\, \varphi = 0$ & $\varphi = \frac{1}{2}\pi$; $A = -\frac{1}{4}$,

$B = + \frac{1}{4}$; unde fit terminus generalis $= (-\frac{1}{4} \text{fin.} \frac{1}{2} \times (n+1)\pi + \frac{1}{4} \text{fin.} \frac{1}{2} n\pi) z^n$. Quare colligendo erit terminus generalis quæſitus $= (\frac{3}{4} n + \frac{9}{8}) z^n \pm \frac{1}{8} z^n - \frac{1}{4} (\text{fin.} \frac{1}{2}(n+1)\pi - \text{fin.} \frac{1}{2} n\pi) z^n$. Hinc

ſi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4} n + 1) z^n$
$n = 4m + 1$	$(\frac{3}{4} n + \frac{5}{4}) z^n$
$n = 4m + 2$	$(\frac{3}{4} n + \frac{3}{2}) z^n$
$n = 4m + 3$	$(\frac{3}{4} n + \frac{3}{4}) z^n$.

Ita, ſi $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39 z^{50}$.

224. Propoſita ergo Serie recurrente, quoniam illa fractio unde oritur, facile cognoſcitur, ejus terminus generalis ſecundum præcepta data reperietur. Ex lege autem Seriei recurrentis, qua quiſque terminus ex præcedentibus definitur, ſtatim innoteſcit denominator fractionis, hujuſque Factores præbebunt formam termini generalis, per numeratorem enim tantum coëfficientes determinantur. Sit nempe propoſita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus lex progreſſionis, qua unuſquiſque terminus ex aliquot præcedentibus determinatur, præbeat hunc fractionis denominatorem $1 - \alpha z - \beta z^2 - \gamma z^3$. Ita ut ſit $D = \alpha C + \beta B + \gamma A$; $E = \alpha D + \beta C + \gamma B$; $F = \alpha E + \beta D + \gamma C$;

 &c., qui multiplicatores α, $+\beta$, $+\gamma$ a Moivræo *ſcalam relationis* conſtituere dicuntur. Lex ergo progreſſionis poſita eſt in ſcala relationis, atque ſcala relationis ſtatim præbet denominatorem fractionis, ex cujus reſolutione propoſita Series recurrens oritur.

225. Ad terminum ergo generalem, ſeu coëfficientem Poteſtatis indefinitæ z^n, inveniendum, quæri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ Factores vel ſimplices vel duplices, ſi imaginarios vitare velimus. Sint primo Factores ſimplices omnes inter ſe inæquales & reales hi $(1-pz)(1-qz)(1-rz)$; atque fractio generans Seriem propoſitam reſolvetur in $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujuſmodi erit $((An + B)p^n + Cr^n)z^n$, &, ſi inſuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod ſi vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat Factorem, ut ſit $= (1-pz)(1 - 2qz.\,coſ.\varphi + qqzz)$ tum terminus generalis erit $= (Ap^n + \frac{B\,ſin.(n+1)\varphi + C\,ſin.\,n\varphi}{ſin.\,\varphi} q^n)z^n$. Cum igitur, poſitis pro n ſucceſſive numeris 0, 1, 2, prodire debeant termini A, Bz, Cz^2, hinc valores litterarum A, B, C determinabuntur.

226. Sit ſcala relationis bimembris, ſeu determinetur quiſque terminus per duos præcedentes, ita ut ſit

$$C = \alpha B - \beta A;\ D = \alpha C - \beta B;\ E = \alpha D - \beta C,\ \&c.,$$

atque manifeſtum eſt Seriem hanc recurrentem, quæ ſit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots\dots\dots + Pz^n + Qz^{n+1} + \&c.,$$

oriri ex fractione cujus denominator sit $1 - \alpha z + \beta zz$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$, erit $p + q = \alpha$ & $pq = \beta$: atque Seriei terminus generalis erit $(Ap^n + Bq^n) z^n$. Hinc facto $n = 0$, erit $A = \mathrm{A} + \mathrm{B}$; & facto $n = 1$, erit $B = \mathrm{A}p + \mathrm{B}q$; unde fit $Aq - B = \mathrm{A}(q - p)$ & $\mathrm{A} = \frac{Aq - B}{q - p}$; & $\mathrm{B} = \frac{Ap - B}{p - q}$. Inventis autem valoribus A & B, erit $P = \mathrm{A}p^n + \mathrm{B}q^n$ & $Q = \mathrm{A}p^{n+1} + \mathrm{B}q^{n+1}$. Tum vero erit $\mathrm{AB} = \frac{BB - \alpha AB + \beta AA}{4\beta - \alpha\alpha}$.

227. Hinc deduci potest modus quemvis terminum ex unico præcedente formandi, cum ad hoc per legem progressionis duo requirantur. Cum enim sit

$$P = \mathrm{A}p^n + \mathrm{B}q^n \ \& \ Q = \mathrm{A}p.p^n + \mathrm{B}q.q^n$$

erit

$$Pq - Q = \mathrm{A}(q - p)p^n \ \& \ Pp - Q = \mathrm{B}(p - q)q^n:$$

multiplicentur hæ expressiones in se invicem; eritque

$$P^2 pq - (p + q)PQ + QQ + \mathrm{AB}(p + q)^2 p^n q^n = 0.$$

At est

$p + q = \alpha$; $pq = \beta$; $(p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 4\beta$ & $p^n q^n = \beta^n$. Quibus substitutis habebitur $\beta P^2 - \alpha PQ + QQ = (\beta AA + \alpha AB - BB)\beta^n$, seu $\frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA} = \beta^n$; quæ est insignis proprietas Serierum recurrentium, quarum quisque terminus per duos præcedentes determinatur. At cognito quovis termino P, erit sequens $Q = \frac{1}{2}\alpha P + \sqrt{((\frac{1}{4}\alpha^2 - \beta)P^2 + (B^2 - \alpha AB + \beta AA)\beta^n)}$, quæ expressio, etsi speciem irrationalitatis præ se

 fert, tamen semper est rationalis, propterea quod termini irrationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis Pz^n & Qz^{n+1} commode assignari potest terminus multo magis remotus Xz^{2n}. Ponatur enim

$X = fP^2 + gPQ - hAB\beta^n$. Quoniam est

$P = Ap^n + Bq^n$ & $Q = Ap.p^n + Bq.q^n$ atque

$X = Ap^{2n} + Bq^{2n}$; erit ut sequitur

$$\begin{aligned} fP^2 &= fA^2p^{2n} + fB^2q^{2n} + 2fAB\beta^n \\ gPQ &= gA^2p.p^{2n} + gB^2q.q^{2n} + gAB\alpha\beta^n \\ -hAB\beta^n &= \qquad\qquad\qquad\qquad -hAB\beta^n \\ \hline X &= Ap^{2n} + Bq^{2n}. \end{aligned}$$

Fiet ergo $f + gp = \frac{1}{A}$; $f + gq = \frac{1}{B}$ & $h = 2f + g\alpha$, unde $g = \frac{B - A}{AB(p-q)}$ & $f = \frac{Ap - Bq}{AB(p-q)}$. At est $B - A = \frac{\alpha A - 2B}{p - q}$; $Ap - Bq = \frac{\alpha B - 2A\beta}{p - q}$. Ergo $f = \frac{\alpha B - 2A\beta}{AB(\alpha\alpha - 4\beta)}$ & $g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\beta)}$ seu $f = \frac{2A\beta - \alpha B}{BB - \alpha AB + \beta AA}$ & $g = \frac{2B - \alpha A}{BB - \alpha AB + \beta AA}$; ideoque $h = \frac{(4\beta - \alpha\alpha)A}{BB - \alpha AB + \beta AA}$.

Eritque ergo

$$X = \frac{(2A\beta - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \beta AA} - A\beta^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\beta A - (\alpha\alpha - 2\beta)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \beta AA)} - \frac{2B\beta^n}{\alpha}.$$

His conjungendis per eliminationem termini β^n reperitur

$$X = \frac{(\beta A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \beta AA}.$$

229. Simili

229. Simili modo, si statuantur termini sequentes

$$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \ldots\ldots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2},$$

erit

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}, \text{ \&, ob } R = \alpha Q - \beta P,$$

erit

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}.$$

At est

$Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$Y = \frac{-\beta BP + 2\beta APQ + \alpha(B - \alpha A)QQ}{BB - \alpha AB + \beta AA}$. Sic igitur porro ex X & Y definiri poterunt simili modo coëfficientes potestatum z^{4n} & z^{4n+1}; hincque ipsarum z^{8n}, z^{8n+1}, & ita porro.

EXEMPLUM.

Sit proposita ista Series recurrens

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \ldots\ldots + Pz^n + Qz^{n+1} + \text{\&c.},$$

cujus cum quilibet coëfficiens sit summa duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \beta AA = 5$; ex quo orietur primum $Q = \frac{P + \surd(5PP + 20(-1)^n)}{2} = \frac{P + \surd(5PP \pm 20)}{2}$, ubi signum superius valet, si n sit numerus par, inferius si impar. Sic, si $n = 4$, ob $P = 11$, erit

 $Q = \frac{11 + \sqrt{(5.121 + 20)}}{2} = \frac{11 + 25}{2} = 18$. Si porro coëfficiens termini z^{2n} sit X, erit $X = \frac{-4PP + 5PQ - QQ}{5}$; ergo Potestatis z^8 coëfficiens erit $= \frac{-4.121 + 6.198 - 324}{5} = 76$. Cum autem sit $Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit $QQ = \frac{3PP \pm 10 + P\sqrt{(5PP \pm 20)}}{2}$; ideoque $X = \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2}$. Ex termino ergo Seriei quocunque Pz^n, obtinentur hi $\frac{P + \sqrt{(5PP \pm 20)}}{2} z^{n+1}$ & $\frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim Series hujusmodi recurrens

$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.$,

cujus scala relationis sit $\alpha, -\beta, +\gamma$, seu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + \beta z^2 - \gamma z^3$. Quod si jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui sint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut sit $P = \mathrm{A}p^n + \mathrm{B}q^n + \mathrm{C}r^n$, $Q = \mathrm{A}p.p^n + \mathrm{B}q.q^n + \mathrm{C}r.r^n$ & $R = \mathrm{A}p^2.p^n + \mathrm{B}q^2.q^n + \mathrm{C}r^2.r^n$; ob $p + q + r = \alpha$, $pq + pr + qr = \beta$ & $pqr = \gamma$, reperietur hæc proportio

$$R^3 \left.\begin{matrix} -2\alpha Q \\ +\beta P \end{matrix}\right\} R^2 \left.\begin{matrix} +(\alpha\alpha + \beta)\,Q^2 \\ -(\alpha\beta + 3\gamma)\,PQ \\ +\alpha\gamma\,P^2 \end{matrix}\right\} R \begin{matrix} -(\alpha\beta - \gamma)\,Q^3 \\ +(\alpha\gamma + \beta\beta)\,PQ^2 \\ -2\beta\gamma\,P^2Q \\ +\gamma\gamma\,P^3 \end{matrix} : \gamma^n =$$

$$C^3 \left.\begin{array}{l} -2\alpha B \\ +\beta A \end{array}\right\} C^2 \left.\begin{array}{l} +(\alpha^2+\beta)B^2 \\ -(\alpha\beta+3\gamma)AB \\ +\alpha\gamma A^2 \end{array}\right\} C \begin{array}{l} -(\alpha\beta-\gamma)B^3 \\ +(\alpha\gamma+\beta\beta)AB^2 \\ -2\beta\gamma A^2B \\ +\gamma\gamma A^3 \end{array} = 0.$$

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a resolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis, superest ut earumdem Serierum summas investigemus. Ac primo quidem manifestum est summam Seriei recurrentis in infinitum extensæ æqualem esse fractioni ex qua oritur: cujus fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est ut numeratorem definiamus. Sit itaque proposita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.;$$

cujus lex progressionis præbeat hunc denominatorem $1-\alpha z+\beta z^2-\gamma z^3+\delta z^4$. Sumamus fractionem summæ Seriei in infinitum æqualem esse $= \frac{a+bz+cz^2+dz^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$, ex qua cum Series proposita oriri debeat, erit comparando (63).

$$\begin{aligned} a &= A \\ b &= B - \alpha A \\ c &= C - \alpha B + \beta A \\ d &= D - \alpha C + \beta B - \gamma A: \end{aligned}$$

Hinc erit summa quæsita

$$\frac{A+(B-\alpha A)z+(C-\alpha B+\beta A)z^2+}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$
$$\frac{(D-\alpha C+\beta B-\gamma A)z^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}.$$

LIB. I.

232. Hinc facile intelligitur quemadmodum Seriei recurrentis summa ad datum terminum usque inveniri debeat. Quæratur scilicet Seriei modo assumtæ summa ad terminum Pz^n, atque ponatur

$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n$,

quoniam hujus Seriei summa in infinitum constat, quæratur summa terminorum ultimum Pz^n in infinitum sequentium, qui sint

$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \&c.$,

hæc Series per z^{n+1} divisa dat Seriem recurrentem propositæ æqualem, cujus propterea summa erit $t =$

$$\frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur summa quæsita $s =$

$$\frac{+A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3 - Qz^{n+1} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(R - \alpha Q)z^{n+2} - (S - \alpha R + \beta Q)z^{n+3} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

233. Quod si ergo scala relationis fuerit bimembris $\alpha, -\mathfrak{C}$; Seriei $A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n$, quæ oritur ex fractione $\frac{A + (B - \alpha A)z}{1 - \alpha z + \mathfrak{C}zz}$, summa erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \mathfrak{C}zz}.$$

At est, ex natura Seriei, $R = \alpha Q - \mathfrak{C}P$, unde prodibit summa

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \mathfrak{C}Pz^{n+2}}{1 - \alpha z + \mathfrak{C}zz}.$$

EXEMPLUM.

Sit proposita Series $1 + 3z + 4z^2 + 7z^3 + \ldots\ldots + Pz^n$ ubi est $\alpha = 1$; $\mathfrak{C} = -1$; $A = 1$; $B = 3$; erit hujus summa $\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz}$. Posito vero $z = 1$, erit summa Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots\ldots + P = P + Q - 3$. Summa ergo termini ultimi & sequentis ternario excedit summam Seriei. Quia vero est $Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit summa Seriei $1 + 3 + 4 + 7 + 11 + \ldots + P = \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}$. Ex solo ergo termino ultimo summa potest exhiberi.

CAPUT XIV.

De multiplicatione ac divisione Angulorum.

234. Sit Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$, ejus Sinus $= x$, Cosinus $= y$ & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti supra vidimus, tam Sinus quam Cosinus Angulorum z; $2z$; $3z$; $4z$; $5z$; &c., constituant Seriem recurrentem cujus scala relationis est $2y, - 1$; primum Sinus horum Arcuum ita se habebunt:

$$\begin{aligned}
fin.\ 0z &= 0\\
fin.\ 1z &= x\\
fin.\ 2z &= 2xy\\
fin.\ 3z &= 4xy^2 - x\\
fin.\ 4z &= 8xy^3 - 4xy\\
fin.\ 5z &= 16xy^4 - 12xy^2 + x\\
fin.\ 6z &= 32xy^5 - 32xy^3 + 6xy\\
fin.\ 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x\\
fin.\ 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 8xy
\end{aligned}$$

hinc concluditur fore

$$fin.\ nz = x\left(2^{n-1}y^{n-1} - (n-2)2^{n-3}y^{n-3} + \frac{(n-3)(n-4)}{1.\ 2}2^{n-5}y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.\ 2.\ 3}2^{n-7}y^{n-7} + \frac{(n-5)(n-6)(n-7)(n-8)}{1.\ 2.\ 3.\ 4}2^{n-9}y^{n-9} - \&c.\right)$$

235. Si ponamus Arcum $nz = s$; erit $fin.\ nz = fin.\ s = fin.\ (\pi - s) = fin.\ (2\pi + s) = fin.\ (3\pi - s)$ &c.,

hi enim Sinus omnes sunt inter se æquales. Hinc obtinemus plures valores pro x, qui erunt CAP. XIV.

$\sin. \frac{s}{n}$; $\sin. \frac{\pi - s}{n}$; $\sin. \frac{2\pi + s}{n}$; $\sin. \frac{3\pi - s}{n}$; $\sin. \frac{4\pi + s}{n}$; &c.,

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo est, ne valores æquales pro iisdem habeantur, quod fiet dum alternæ tantum expressiones assumantur. Cognitis igitur radicibus æquationis a posteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita insit, requiritur, pro y suus valor $\sqrt{(1 - xx)}$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $-z, +z, +3z, +5z$, &c., differentia est $2z$, hujusque Cosinus $= 1 - 2xx$, erit progressionis Sinuum scala relationis hæc $2 - 4xx, - 1$. Hinc erit

$$\begin{aligned}
\sin. -z &= -x \\
\sin. \; z &= x \\
\sin. \; 3z &= 3x - 4x^3 \\
\sin. \; 5z &= 5x - 20x^3 + 16x^5 \\
\sin. \; 7z &= 7x - 56x^3 + 112x^5 - 64x^7 \\
\sin. \; 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9
\end{aligned}$$

ergo

$$\sin. nz = nx - \frac{n(nn-1)}{1.\,2.\,3} x^3 + \frac{n(nn-1)(nn-9)}{1.\,2.\,3.\,4.\,5} x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.\,2.\,3.\,4.\,5.\,6.\,7} x^7 + \text{\&c.},$$

LIB. I. si quidem n fuerit numerus impar. Hujusque æquationis radices sunt $fin.\, z$; $fin.\, (\frac{2\pi}{n} + z)$; $fin.\, (\frac{4\pi}{n} + z)$; $fin.\, (\frac{6\pi}{n} + z)$; $fin.\, (\frac{8\pi}{n} + z)$; &c., quarum numerus est n.

237. Hujus ergo æquationis

$$0 = 1 - \frac{nx}{fin.\, nz} + \frac{n(nn-1)x^3}{1.2.3\, fin.\, nz} - \frac{n(nn-1)(nn-9)x^5}{1.2.3.4.5\, fin.\, nz} \ldots \pm \frac{2^{n-1}x^n}{fin.\, nz},$$

(ubi signum superius valet si n unitate deficiat a multiplo quaternarii, contra inferius) Factores sunt $(1 - \frac{x}{fin.\, z})$ $(1 - \frac{x}{fin.\, (\frac{2\pi}{n} + z)})$ $(1 - \frac{x}{fin.\, (\frac{4\pi}{n} + z)})$ &c., ex quibus concluditur fore

$$\frac{n}{fin.\, nz} = \frac{1}{fin.\, z} + \frac{1}{fin.\, (\frac{2\pi}{n} + z)} + \frac{1}{fin.\, (\frac{4\pi}{n} + z)} + \frac{1}{fin.\, (\frac{6\pi}{n} + z)} +$$

&c., donec habeantur n termini. Tum vero productum omnium erit

$$\mp \frac{2^{n-1}}{fin.\, nz} = \frac{1}{fin.\, z.\, fin.\, (\frac{2\pi}{n} + z)\, fin.\, (\frac{4\pi}{n} + z)\, fin.\, (\frac{6\pi}{n} + z)\, \&c.}$$

seu $fin.\, nz = \mp 2^{n-1} fin.\, z.\, fin.\, (\frac{2\pi}{n} + z)\, fin.\, (\frac{4\pi}{n} + z) \times fin.\, (\frac{6\pi}{n} + z)$ &c.. Et, quia terminus penultimus deest, erit

$$0 = fin.\, z + fin.\, (\frac{2\pi}{n} + z) + fin.\, (\frac{4\pi}{n} + z) + fin.\, (\frac{6\pi}{n} + z)\, \&c.$$

EXEMPLUM I.

Si ergo fuerit $n = 3$, prodibunt hæ æqualitates $0 = fin.\, z + fin.\, (120° + z) + fin.\, (240° + z) = fin.\, z + fin.\, (60° - z) - fin.\, (60° + z)$.

$\frac{3}{fin.\, 3z}$

$\frac{3}{\sin. 3z} = \frac{1}{\sin. z} + \frac{1}{\sin. (120° + z)} + \frac{1}{\sin. (240° + z)} = \frac{1}{\sin. z} + \frac{1}{\sin. (60° - z)} - \frac{1}{\sin. (60° + z)}$ CAP. XIV.

$\sin. 3z = -4 \sin. z. \sin. (120° + z) \sin. (240° + z) = 4 \sin. z. \sin. (60° - z). \sin. (60° + z)$.

Erit ergo, uti jam ſupra notavimus,

$\sin. (60° + z) = \sin. z + \sin. (60° - z)$ &

$3 \operatorname{coſec.} 3z = \operatorname{coſec.} z + \operatorname{coſec.} (60° - z) - \operatorname{coſec.} (60° + z)$.

EXEMPLUM II.

Ponamus eſſe $n = 5$, atque prodibunt hæ æquationes:

$0 = \sin. z + \sin. (\frac{2}{5} \pi + z) + \sin. (\frac{4}{5} \pi + z) + \sin. (\frac{6}{5} \pi + z) + \sin. (\frac{8}{5} \pi + z)$

ſeu $0 = \sin. z + \sin. (\frac{2}{5} \pi + z) + \sin. (\frac{1}{5} \pi - z) - \sin. (\frac{1}{5} \pi + z) - \sin. (\frac{2}{5} \pi - z)$

ſeu $0 = \sin. z + \sin. (\frac{1}{5} \pi - z) - \sin. (\frac{1}{5} \pi + z) + \sin. (\frac{2}{5} \pi + z) - \sin. (\frac{2}{5} \pi - z)$

deinde erit

$\frac{5}{\sin. 5z} = \frac{1}{\sin. z} + \frac{1}{\sin. (\frac{1}{5} \pi - z)} - \frac{1}{\sin. (\frac{1}{5} \pi + z)} - \frac{1}{\sin. (\frac{2}{5} \pi - z)} + \frac{1}{\sin. (\frac{2}{5} \pi + z)}$

$\sin. 5z = 16 \sin. z. \sin. (\frac{1}{5} \pi - z) \sin. (\frac{1}{5} \pi + z) \times \sin. (\frac{2}{5} \pi - z) \sin. (\frac{2}{5} \pi + z)$.

EXEMPLUM III.

Hoc modo, si ponamus $n = 2m + 1$, erit

$$0 = \sin. z + \sin. (\tfrac{\pi}{n} - z) - \sin. (\tfrac{\pi}{n} + z) - \sin. (\tfrac{2\pi}{n} - z) + \sin. (\tfrac{2\pi}{n} + z) + \sin. (\tfrac{3\pi}{n} - z) - \sin. (\tfrac{3\pi}{n} + z) - \ldots \pm \sin. (\tfrac{m}{n}\pi - z) \mp \sin. (\tfrac{m}{n}\pi + z)$$

ubi signa superiora valent si m sit numerus impar, inferiora si sit par. Altera æquatio erit hæc

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin. (\frac{\pi}{n} - z)} - \frac{1}{\sin. (\frac{\pi}{n} + z)} - \frac{1}{\sin. (\frac{2\pi}{n} - z)} + \frac{1}{\sin. (\frac{2\pi}{n} + z)} + \frac{1}{\sin. (\frac{3\pi}{n} - z)} - \frac{1}{\sin. (\frac{3\pi}{n} + z)} - \ldots \pm \frac{1}{\sin. (\frac{m\pi}{n} - z)} \mp \frac{1}{\sin. (\frac{m\pi}{n} + z)}$$

quæ ad Cosecantes commode transfertur. Tertio habetur hoc productum:

$$\sin. nz = 2^{2m} \sin. z. \sin. (\tfrac{\pi}{n} - z) \sin. (\tfrac{\pi}{n} + z) \sin. (\tfrac{2\pi}{n} - z) \times \sin. (\tfrac{2\pi}{n} + z) \sin. (\tfrac{3\pi}{n} - z) \sin. (\tfrac{3\pi}{n} + z) \ldots \ldots \times \sin. (\tfrac{m\pi}{n} - z) \sin. (\tfrac{m\pi}{n} + z).$$

238. Sit n nunc numerus par, & quoniam est $y = \surd(1 - xx)$ & $cos. 2z = 1 - 2xx$, ita ut Seriei Sinuum sit scala relationis, ut ante, $2 - 4xx, -1$, erit

$$\sin. 0z = 0$$
$$\sin. 2z = 2x\sqrt{(1-xx)}$$
$$\sin. 4z = (4x - 8x^3)\sqrt{(1-xx)}$$
$$\sin. 6z = (6x - 32x^3 + 32x^5)\sqrt{(1-xx)}$$
$$\sin. 8z = (8x - 80x^3 + 192x^5 - 128x^7)\sqrt{(1-xx)}$$

& generaliter

$$\sin. nz = \left(nx - \frac{n(nn-4)}{1.\,2.\,3}x^3 + \frac{n(nn-4)(nn-16)}{1.\,2.\,3.\,4.\,5}x^5 - \frac{n(nn-4)(nn-16)(nn-36)}{1.\,2.\,3.\,4.\,5.\,6.\,7}x^7 + \ldots\ldots \pm 2^{n-1}x^{n-1}\right)\sqrt{(1-xx)}$$

denotante n numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam ſumantur utrinque quadrata, ac prodibit hujuſmodi æquatio

$$(\sin. nz)^2 = nnxx + Px^4 + Qx^6 + \ldots\ldots - 2^{2n-2}x^{2n}$$

ſeu
$$x^{2n}\ldots\ldots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}(\sin. nz)^2 = 0$$

cujus æquationis radices erunt tam affirmativæ quam negativæ; Scilicet $\pm \sin. z$; $\pm \sin.(\frac{\pi}{n} - z)$; $\pm \sin.(\frac{2\pi}{n} + z)$; $\pm \sin.(\frac{3\pi}{n} - z)$; $\pm \sin.(\frac{4\pi}{n} + z)$ &c. Sumendo omnino n hujuſmodi expreſſiones. Cum igitur ultimus terminus ſit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin. nz = \pm 2^{n-1}\sin. z.\sin.\left(\frac{\pi}{n} - z\right)\sin.\left(\frac{2\pi}{n} + z\right)\times \sin.\left(\frac{3\pi}{n} - z\right)\ldots\ldots;$$

ubi, quibus caſibus utrumvis ſignum valeat, ex caſibus particularibus erit diſpiciendum.

EXEMPLUM.

Subſtituendo autem pro n ſucceſſive numeros 2, 4, 6, &c. & eligendo n Sinus diverſos erit.

$$\sin. 2z = 2\sin. z. \sin.(\frac{\pi}{2} - z)$$

$$\sin. 4z = 8\sin. z. \sin.(\frac{\pi}{4} - z)\sin.(\frac{\pi}{4} + z)\sin.(\frac{\pi}{2} - z)$$

$$\sin. 6z = 32\sin. z. \sin.(\frac{\pi}{6} - z)\sin.(\frac{\pi}{6} + z)\sin.(\frac{2\pi}{6} - z)\times \sin.(\frac{2\pi}{6} + z)\sin.(\frac{3\pi}{6} - z)$$

$$\sin. 8z = 128\sin. z. \sin.(\frac{\pi}{8} - z)\sin.(\frac{\pi}{8} + z)\sin.(\frac{2\pi}{8} - z)\times \sin.(\frac{2\pi}{8} + z)\sin.(\frac{3\pi}{8} - z)\sin.(\frac{3\pi}{8} + z)\sin.(\frac{4\pi}{8} - z).$$

240. Patet ergo fore generatim

$$\sin. nz = 2^{n-1}\sin. z. \sin.(\frac{\pi}{n} - z)\sin.(\frac{\pi}{n} + z.)\sin.(\frac{2\pi}{n} - z)\times \sin.(\frac{2\pi}{n} + z)\sin.(\frac{3\pi}{n} - z)\sin.(\frac{3\pi}{n} + z)\ldots\sin.(\frac{1}{2}\pi - z)$$

si n fuerit numerus par. Quod si autem hæc cum superiori, ubi n erat numerus impar, comparetur, tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\sin. nz = 2^{n-1}\sin. z. \sin.(\frac{\pi}{n} - z).\sin.(\frac{\pi}{n} + z)\sin.(\frac{2\pi}{n} - z)\times \sin.(\frac{2\pi}{n} + z)\sin.(\frac{3\pi}{n} - z)\sin.(\frac{3\pi}{n} + z) \&c.$$

donec tot habeantur Factores, quot numerus n continet unitates.

241. Expressiones istæ, quibus Sinus Angulorum multiplorum per Factores exponuntur, non parum utilitatis afferre possunt ad Logarithmos Sinuum Angulorum multiplorum inveniendos, itemque ad plures expressiones Sinuum per Factores, quales supra (§. 184.) dedimus, reperiendas. Erit autem

$$\sin. z = 1 \sin. z$$
$$\sin. 2z = 2 \sin. z. \sin. (\tfrac{\pi}{2} - z)$$
$$\sin. 3z = 4 \sin. z. \sin. (\tfrac{\pi}{3} - z) \sin. (\tfrac{\pi}{3} + z)$$
$$\sin. 4z = 8 \sin. z. \sin. (\tfrac{\pi}{4} - z) \sin. (\tfrac{\pi}{4} + z) \sin. (\tfrac{2\pi}{4} - z)$$
$$\sin. 5z = 16 \sin. z. \sin. (\tfrac{\pi}{5} - z) \sin. (\tfrac{\pi}{5} + z) \sin. (\tfrac{2\pi}{5} - z) \times \sin. (\tfrac{2\pi}{5} + z)$$
$$\sin. 6z = 32 \sin. z. \sin. (\tfrac{\pi}{6} - z) \sin. (\tfrac{\pi}{6} + z) \sin. (\tfrac{2\pi}{6} - z) \times \sin. (\tfrac{2\pi}{6} + z) \sin. (\tfrac{3\pi}{6} - z)$$

&c.

242. Cum deinde sit $\frac{\sin. 2nz}{\sin. nz} = 2 \cos. nz$, Cosinus Angulorum multiplorum simili modo per Factores exprimentur:

$$\cos. z = 1 \sin. (\tfrac{\pi}{2} - z)$$
$$\cos. 2z = 2 \sin. (\tfrac{\pi}{4} - z) \sin. (\tfrac{\pi}{4} + z)$$
$$\cos. 3z = 4 \sin. (\tfrac{\pi}{6} - z) \sin. (\tfrac{\pi}{6} + z) \sin. (\tfrac{3\pi}{6} - z)$$
$$\cos. 4z = 8 \sin. (\tfrac{\pi}{8} - z) \sin. (\tfrac{\pi}{8} + z) \sin. (\tfrac{3\pi}{8} - z) \times \sin. (\tfrac{3\pi}{8} + z)$$
$$\cos. 5z = 16 \sin. (\tfrac{\pi}{10} - z) \sin. (\tfrac{\pi}{10} + z) \sin. (\tfrac{3\pi}{10} - z) \times \sin. (\tfrac{3\pi}{10} + z) \sin. (\tfrac{5\pi}{10} - z)$$

& generaliter

LIB. I. $cos.\, nz = 2^{n-1} \sin.(\frac{\pi}{2n} - z)\, \sin.(\frac{\pi}{2n} + z)\, \sin.(\frac{3\pi}{2n} - z) \times$
$\sin.(\frac{3\pi}{2n} + z)\, \sin.(\frac{5\pi}{2n} - z)$ &c.,

quoad tot habeantur Factores quot numerus n continet unitates.

243. Eædem expressiones prodibunt ex consideratione Cosinuum Arcuum multiplorum; si enim fuerit $cos.\, z = y$, erit ut sequitur

$$\begin{aligned}
cos.\, 0z &= 1 \\
cos.\, 1z &= y \\
cos.\, 2z &= 2yy - 1 \\
cos.\, 3z &= 4y^3 - 3y \\
cos.\, 4z &= 8y^4 - 8yy + 1 \\
cos.\, 5z &= 16y^5 - 20y^3 + 5y \\
cos.\, 6z &= 32y^6 - 48y^4 + 18yy - 1 \\
cos.\, 7z &= 64y^7 - 112y^5 + 56y^3 - 7y
\end{aligned}$$

& generaliter

$$cos.\, nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1.\ 2} \times 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1.\ 2.\ 3} 2^{n-7} y^{n-6} + \frac{n(n-5)(n-6)(n-7)}{1.\ 2.\ 3.\ 4} 2^{n-9} y^{n-8} - \&c.,$$

cujus æquationis, cum sit $cos.\, nz = cos.\,(2\pi - nz) = cos.\,(2\pi + nz) = cos.\,(4\pi \pm nz) = cos.\,(6\pi \pm nz)$ &c., erunt radices ipsius y hæ: $cos.\, z$; $cos.\,(\frac{2\pi}{n} \pm z)$; $cos.\,(\frac{4\pi}{n} \pm z)$; $cos.\,(\frac{6\pi}{n} \pm z)$ &c., quarum formularum tot diversæ sunt pro y eligendæ quot dantur; dantur autem tot, quot n continet unitates.

244. Primum igitur patet, ob terminum secundum deficientem excepto casu $n = 1$, fore summam harum radicum omnium $= 0$. Erit ergo

$$0 = \cos. z + \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right) + \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right) + \&c.,$$

sumendo tot terminos quot n continet unitates: hæc autem æqualitas sponte se offert si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares, unitate exclusa, eritque, ob $\cos. v = -\cos.(\pi - v)$,

$$0 = \cos. z - \cos.\left(\frac{\pi}{3} - z\right) - \cos.\left(\frac{\pi}{3} + z\right)$$

$$0 = \cos. z - \cos.\left(\frac{\pi}{5} - z\right) - \cos.\left(\frac{\pi}{5} + z\right) + \cos.\left(\frac{2\pi}{5} - z\right) + \cos.\left(\frac{2\pi}{5} + z\right)$$

$$0 = \cos. z - \cos.\left(\frac{\pi}{7} - z\right) - \cos.\left(\frac{\pi}{7} + z\right) + \cos.\left(\frac{2\pi}{7} - z\right) + \cos.\left(\frac{2\pi}{7} + z\right) - \cos.\left(\frac{3\pi}{7} - z\right) - \cos.\left(\frac{3\pi}{7} + z\right)$$

& generaliter, si fuerit n numerus impar quicunque, erit

$$0 = \cos. z - \cos.\left(\frac{\pi}{n} - z\right) - \cos.\left(\frac{\pi}{n} + z\right) + \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right) - \cos.\left(\frac{3\pi}{n} - z\right) - \cos.\left(\frac{3\pi}{n} + z\right) + \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right) - \&c.,$$

sumendo tot terminos, quot numerus n continet unitates: oportet autem n esse numerum imparem unitate majorem, uti jam monuimus.

LIB. I. 245. Quod ad productum ex omnibus attinet, variæ quidem prodeunt expressiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par: omnes autem comprehenduntur in expressione generali (§. 242) inventa, si singuli Sinus in Cosinus transmutentur: erit scilicet

$$cos.\ z = 1\ cos.\ z$$

$$cos.\ 2z = 2\ cos.\left(\tfrac{\pi}{4} + z\right) cos.\left(\tfrac{\pi}{4} - z\right)$$

$$cos.\ 3z = 4\ cos.\left(\tfrac{2\pi}{6} + z\right) cos.\left(\tfrac{2\pi}{6} - z\right) cos.\ z$$

$$cos.\ 4z = 8\ cos.\left(\tfrac{3\pi}{8} + z\right) cos.\left(\tfrac{3\pi}{8} - z\right) cos.\left(\tfrac{\pi}{8} + z\right) \times cos.\left(\tfrac{\pi}{8} - z\right)$$

$$cos.\ 5z = 16\ cos.\left(\tfrac{4\pi}{10} + z\right) cos.\left(\tfrac{4\pi}{10} - z\right) cos.\left(\tfrac{2\pi}{10} + z\right) \times cos.\left(\tfrac{2\pi}{10} - z\right) cos.\ z;$$

& generaliter

$$cos.\ nz = 2^{n-1}\ cos.\left(\tfrac{n-1}{2n}\pi + z\right) cos.\left(\tfrac{n-1}{2n}\pi - z\right) \times cos.\left(\tfrac{n-3}{2n}\pi + z\right) cos.\left(\tfrac{n-3}{2n}\pi - z\right) \times cos.\left(\tfrac{n-5}{2n}\pi + z\right) cos.\left(\tfrac{n-5}{2n}\pi - z\right) \times cos.\left(\tfrac{n-7}{2n}\pi + z\right) \&c.,$$

sumtis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit $0 = 1 \mp \frac{ny}{cos.\ nz} + \&c.$, ubi signum superius valet si n fuerit numerus impar formæ $4m + 1$, inferius si $n = 4m - 1$. Hinc erit

+

$$+\frac{1}{cos.\,z}=\frac{1}{cos.\,z}$$

$$-\frac{3}{cos.\,3z}=\frac{1}{cos.\,z}-\frac{1}{cos.(\frac{\pi}{3}-z)}-\frac{1}{cos.(\frac{\pi}{3}+z)}$$

$$+\frac{5}{cos.\,5z}=\frac{1}{cos.\,z}-\frac{1}{cos.(\frac{\pi}{5}-z)}-\frac{1}{cos.(\frac{\pi}{5}+z)}+\frac{1}{cos.(\frac{2\pi}{5}-z)}+\frac{1}{cos.(\frac{2\pi}{5}+z)}$$

& generaliter, posito $n=2m+1$, erit

$$\frac{n}{cos.\,nz}=\frac{2m+1}{cos.\,(2m+1)z}=\frac{1}{cos.(\frac{m}{n}\pi+z)}+\frac{1}{cos.(\frac{m}{n}\pi-z)}-\frac{1}{cos.\,(\frac{m-1}{n}\pi+z)}-\frac{1}{cos.\,(\frac{m-1}{n}\pi-z)}+\frac{1}{cos.\,(\frac{m-2}{n}\pi+z)}+\frac{1}{cos.\,(\frac{m-2}{n}\pi-z)}-\frac{1}{cos.\,(\frac{m-3}{n}\pi+z)}-\text{\&c.}$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{cos.\,v}=sec.\,v$, hinc pro Secantibus insignes proprietates deducuntur, erit nempe

$$sec.\ \ z=sec.\,z$$

$$3sec.\,3z=sec.(\frac{\pi}{3}+z)+sec.(\frac{\pi}{3}-z)-sec.(\frac{0\pi}{3}+z)$$

$$5sec.\,5z=sec.(\frac{2\pi}{5}+z)+sec.(\frac{2\pi}{5}-z)-sec.(\frac{\pi}{5}+z)-sec.(\frac{\pi}{5}-z)+sec.(\frac{0\pi}{5}+z)$$

$$7\,sec.\,7z = sec.\left(\tfrac{3\pi}{7}+z\right)+sec.\left(\tfrac{3\pi}{7}-z\right)-sec.\left(\tfrac{2\pi}{7}+z\right)-$$
$$sec.\left(\tfrac{2\pi}{7}-z\right)-sec.\left(\tfrac{\pi}{7}+z\right)+sec.\left(\tfrac{\pi}{7}-z\right)-$$
$$sec.\left(\tfrac{0\pi}{7}+z\right)$$

& generaliter, posito $n = 2m+1$, erit

$$n.\,sec.\,nz = sec.\left(\tfrac{m}{n}\pi+z\right)+sec.\left(\tfrac{m}{n}\pi-z\right)-$$
$$sec.\left(\tfrac{m-1}{n}\pi+z\right)-sec.\left(\tfrac{m-1}{n}\pi-z\right)+$$
$$sec.\left(\tfrac{m-2}{n}\pi+z\right)+sec.\left(\tfrac{m-2}{n}\pi-z\right)-$$
$$sec.\left(\tfrac{m-3}{n}\pi+z\right)-sec.\left(\tfrac{m-3}{n}\pi-z\right)+$$
$$sec.\left(\tfrac{m-4}{n}\pi+z\right)+\ldots\ldots\ldots\ldots\pm sec.\,z.$$

248. Pro Cosecantibus autem erit ex §. 237:

$$cosec.\ z = cosec.\,z$$
$$3\,cosec.\,3z = cosec.\,z + cosec.\left(\tfrac{\pi}{3}-z\right)-cosec.\left(\tfrac{\pi}{3}+z\right)$$
$$5\,cosec.\,5z = cosec.\,z + cosec.\left(\tfrac{\pi}{5}-z\right)-cosec.\left(\tfrac{\pi}{5}+z\right)-$$
$$cosec.\left(\tfrac{2\pi}{5}-z\right)+cosec.\left(\tfrac{2\pi}{5}+z\right)$$
$$7\,cosec.\,7z = cosec.\,z + cosec.\left(\tfrac{\pi}{7}-z\right)-cosec.\left(\tfrac{\pi}{7}+z\right)-$$
$$cosec.\left(\tfrac{2\pi}{7}-z\right)+cosec.\left(\tfrac{2\pi}{7}+z\right)+$$
$$cosec.\left(\tfrac{3\pi}{7}-z\right)-cosec.\left(\tfrac{3\pi}{7}+z\right)$$

& generaliter, ponendo $n = 2m+1$, erit

$$n.\,cosec.\,nz = cosec.\,z + cosec.\,(\frac{\pi}{n} - z) - cosec.\,(\frac{\pi}{n} + z) -$$ CAP. XIV.

$$cosec.\,(\frac{2\pi}{n} - z) + cosec.\,(\frac{2\pi}{n} + z) +$$

$$cosec.\,(\frac{3\pi}{n} - z) - cosec.\,(\frac{3\pi}{n} + z) -$$

$$\ldots\ldots\ldots\ldots \mp cosec.\,(\frac{m\pi}{n} - z) \pm cosec.\,(\frac{m\pi}{n} + z)$$

ubi ſigna ſuperiora valent ſi m fuerit numerus par, inferiora ſi m ſit impar.

249. Cum ſit, uti ſupra vidimus, $cof.\,nz \pm \sqrt{-1}.\,fin.\,nz = (cof.\,z \pm \sqrt{-1}.\,fin.\,z)^n$, erit $cof.\,nz =$

$$\frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n + (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{2}\;;\; fin.\,nz =$$

$$\frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n - (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{2\sqrt{-1}}\;\&$$

$$tang.\,nz = \frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n - (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n\sqrt{-1} + (cof.\,z - \sqrt{-1}.\,fin.\,z)^n\sqrt{-1}}.$$

Ponamus $tang.\,z = \frac{fin.\,z}{cof.\,z} = t$, erit $tang.\,nz =$

$$\frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n\sqrt{-1} + (1 - t\sqrt{-1})^n\sqrt{-1}},$$ unde oriuntur Tangentes Angulorum multiplorum ſequentes

$$tang.\;z = t$$

$$tang.\;2z = \frac{2t}{1 - tt}$$

$$tang.\;3z = \frac{3t - t^3}{1 - 3tt}$$

$$tang.\;4z = \frac{4t - 4t^3}{1 - 6tt + t^4}$$

$$tang.\;5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

& generaliter

$$tang.\,nz = \frac{nt - \frac{n(n-1)(n-2)}{1.\,2.\,3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\,2.\,3.\,4.\,5}t^5 - \&c.}{1 - \frac{n(n-1)}{1.\,2}tt + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}t^4 - \&c.}.$$

Cum jam ſit $tang.\,nz = tang.(\pi + nz) = tang.(2\pi + nz) = tang.(3\pi + nz)$ &c., erunt valores ipſius t, ſeu radices æquationis, hæ, $tang.\,z$; $tang.(\frac{\pi}{n} + z)$; $tang.(\frac{2\pi}{n} + z)$; $tang.(\frac{3\pi}{n} + z)$; &c., quarum numerus eſt n.

250. Quod ſi æquatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{tang.\,nz} - \frac{n(n-1)tt}{1.\,2} + \frac{n(n-1)(n-2)t^3}{1.\,2.\,3\,tang.\,nz} + \&c..$$

Ex comparatione ergo coëfficientium cum radicibus, erit

$$n.\,cot.\,nz = cot.\,z + cot.(\frac{\pi}{n} + z) + cot.(\frac{2\pi}{n} + z) + cot.(\frac{3\pi}{n} + z) + cot.(\frac{4\pi}{n} + z) + \ldots\ldots + cot.(\frac{n-1}{n}\pi + z)$$

deinde erit ſumma quadratorum harum Cotangentium omnium $= \frac{nn}{(fin.\,nz)^2} - n$, ſimilique modo ulteriores Poteſtates poſſunt definiri. Ponendo autem loco n numeros definitos, erit.

$$cot.\,z = cot.\,z$$

$$2\,cot.\,2z = cot.\,z + cot.(\frac{\pi}{2} + z)$$

$$3\,cot.\,3z = cot.\,z + cot.(\frac{\pi}{3} + z) + cot.(\frac{2\pi}{3} + z)$$

$$4\,cot.\,4z = cot.\,z + cot.(\frac{\pi}{4} + z) + cot.(\frac{2\pi}{4} + z) + cot.(\frac{3\pi}{4} + z)$$

$$5 \cot. 5z = \cot. z + \cot. \left(\frac{\pi}{5} + z\right) + \cot. \left(\frac{2\pi}{5} + z\right) + \cot. \left(\frac{3\pi}{5} + z\right) + \cot. \left(\frac{4\pi}{5} + z\right).$$

251. Quia vero eſt $\cot. v = - \cot. (\pi - v)$, erit

$$\cot. \; z = \cot. z$$

$$2 \cot. 2z = \cot. z - \cot. \left(\frac{\pi}{2} - z\right)$$

$$3 \cot. 3z = \cot. z - \cot. \left(\frac{\pi}{3} - z\right) + \cot. \left(\frac{\pi}{3} + z\right)$$

$$4 \cot. 4z = \cot. z - \cot. \left(\frac{\pi}{4} - z\right) + \cot. \left(\frac{\pi}{4} + z\right) - \cot. \left(\frac{2\pi}{4} - z\right)$$

$$5 \cot. 5z = \cot. z - \cot. \left(\frac{\pi}{5} - z\right) + \cot. \left(\frac{\pi}{5} + z\right) - \cot. \left(\frac{2\pi}{5} - z\right) + \cot. \left(\frac{2\pi}{5} + z\right)$$

& generaliter

$$n. \cot. nz = \cot. z - \cot. \left(\frac{\pi}{n} - z\right) + \cot. \left(\frac{\pi}{n} + z\right) - \cot. \left(\frac{2\pi}{n} - z\right) + \cot. \left(\frac{2\pi}{n} + z\right) - \cot. \left(\frac{3\pi}{n} - z\right) + \cot. \left(\frac{3\pi}{n} + z\right) - \&c.$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus æquationem inventam a Poteſtate ſumma, ubi primum diſtinguendi ſunt caſus, quibus n eſt vel numerus par, vel impar. Sit n numerus impar, ſeu $n = 2m + 1$, erit

LIB. I. $t - tang.\, z = 0$

$t^3 - 3tt.\, tang.\, 3z - 3t + tang.\, 3z = 0$

$t^5 - 5t^4.\, tang.\, 5z - 10t^3 + 10tt.\, tang.\, 5z + 5t - tang.\, 5z = 0$

& generaliter

$t^n - nt^{n-1}.\, tang.\, nz - \ldots\ldots\ldots\ldots \mp tang.\, nz = 0$

ubi ſignum ſuperius — valet, ſi *m* ſit numerus par, inferius + ſi *m* ſit numerus impar. Erit ergo ex coëfficiente ſecundi termini

$$tang.\, z = tang.\, z$$

$$3\, tang.\, 3z = tang.\, z + tang.\left(\frac{\pi}{3} + z\right) + tang.\left(\frac{2\pi}{3} + z\right)$$

$$5\, tang.\, 5z = tang.\, z + tang.\left(\frac{\pi}{5} + z\right) + tang.\left(\frac{2\pi}{5} + z\right) + tang.\left(\frac{3\pi}{5} + z\right) + tang.\left(\frac{4\pi}{5} + z\right).$$

&c.

253. Cum igitur ſit $tang.\, v = - tang.\, (\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$$tang.\, z = tang.\, z$$

$$3\, tang.\, 3z = tang.\, z - tang.\left(\frac{\pi}{3} - z\right) + tang.\left(\frac{\pi}{3} + z\right)$$

$$5\, tang.\, 5z = tang.\, z - tang.\left(\frac{\pi}{5} - z\right) + tang.\left(\frac{\pi}{5} + z\right) - tang.\left(\frac{2\pi}{5} - z\right) + tang.\left(\frac{2\pi}{5} + z\right)$$

$$7\, tang.\, 7z = tang.\, z - tang.\left(\frac{\pi}{7} - z\right) + tang.\left(\frac{\pi}{7} + z\right) - tang.\left(\frac{2\pi}{7} - z\right) + tang.\left(\frac{2\pi}{7} + z\right) - tang.\left(\frac{3\pi}{7} - z\right) + tang.\left(\frac{3\pi}{7} + z\right)$$

& generaliter, si $n = 2m + 1$, erit

$$n.\,tang.\,nz = tang.\,z - tang.\left(\frac{\pi}{n} - z\right) + tang.\left(\frac{\pi}{n} + z\right) - tang.\left(\frac{2\pi}{n} - z\right) + tang.\left(\frac{2\pi}{n} + z\right) - tang.\left(\frac{3\pi}{n} - z\right) + \ldots\ldots\ldots\ldots - tang.\left(\frac{m\pi}{n} - z\right) + tang.\left(\frac{m\pi}{n} + z\right).$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.\,nz$, propterea quod per signorum negativorum numerum alternatim parem & imparem, superior signorum ambiguitas tollitur. Sic erit

$$tang.\,z = tang.\,z$$

$$tang.\,3z = tang.\,z.\,tang.\left(\frac{\pi}{3} - z\right).\,tang.\left(\frac{\pi}{3} + z\right)$$

$$tang.\,5z = tang.\,z.\,tang.\left(\frac{\pi}{5} - z\right).\,tang.\left(\frac{\pi}{5} + z\right).\,tang.\left(\frac{2\pi}{5} - z\right) \times tang.\left(\frac{2\pi}{5} + z\right)$$

& generaliter, si $n = 2m + 1$, erit

$$tang.\,nz = tang.\,z.\,tang.\left(\frac{\pi}{n} - z\right).\,tang.\left(\frac{\pi}{n} + z\right).\,tang.\left(\frac{2\pi}{n} - z\right) \times tang.\left(\frac{2\pi}{n} + z\right).\,tang.\left(\frac{3\pi}{n} - z\right) \ldots\ldots\ldots\ldots \times tang.\left(\frac{m\pi}{n} - z\right).\,tang.\left(\frac{m\pi}{n} + z\right).$$

255. Sit jam n numerus par, atque, incipiendo a Potestate summa, erit

$$tt + 2t.\,cot.\,2z - 1 = 0$$

$$t^4 + 4t^3.\,cot.\,4z - 6tt - 4t.\,cot.\,4z + 1 = 0$$

LIB. I. & generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1}.\,cot.\,nz - \ldots\ldots\ldots\ldots \mp 1 = 0$$

ubi signum superius — valet si m sit numerus impar, inferius + si m sit par. Comparando ergo radices cum coëfficiente secundi termini, erit

$$-2\,cot.\,2z = tang.\,z + tang.\left(\frac{\pi}{2} + z\right)$$

$$-4\,cot.\,4z = tang.\,z + tang.\left(\frac{\pi}{4} + z\right) + tang.\left(\frac{2\pi}{4} + z\right) + tang.\left(\frac{3\pi}{4} + z\right)$$

$$-6\,cot.\,6z = tang.\,z + tang.\left(\frac{\pi}{6} + z\right) + tang.\left(\frac{2\pi}{6} + z\right) + tang.\left(\frac{3\pi}{6} + z\right) + tang.\left(\frac{4\pi}{6} + z\right) + tang.\left(\frac{5\pi}{6} + z\right).$$

&c.

256. Cum sit $tang.\,v = -\,tang.\,(\pi - v)$, sequentes formabuntur æquationes

$$2\,cot.\,2z = -\,tang.\,z + tang.\left(\frac{\pi}{2} - z\right)$$

$$4\,cot.\,4z = -\,tang.\,z + tang.\left(\frac{\pi}{4} - z\right) - tang.\left(\frac{\pi}{4} + z\right) + tang.\left(\frac{2\pi}{4} - z\right)$$

$$6\,cot.\,6z = -\,tang.\,z + tang.\left(\frac{\pi}{6} - z\right) - tang.\left(\frac{\pi}{6} + z\right) + tang.\left(\frac{2\pi}{6} - z\right) - tang.\left(\frac{2\pi}{6} + z\right) + tang.\left(\frac{3\pi}{6} - z\right)$$

& generaliter, si $n = 2m$, erit

$$n.\,cot.\,nz = -\,tang.\,z + tang.\,(\tfrac{\pi}{n} - z) - tang.\,(\tfrac{\pi}{n} + z) + tang.\,(\tfrac{2\pi}{n} - z) - tang.\,(\tfrac{2\pi}{n} + z) + tang.\,(\tfrac{3\pi}{n} - z) - tang.\,(\tfrac{3\pi}{n} + z) + \dots\dots\dots\dots + tang.\,(\tfrac{m\pi}{n} - z).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur; eritque idcirco

$$1 = tang.\,z.\,tang.\,(\tfrac{\pi}{2} - z)$$

$$1 = tang.\,z.\,tang.\,(\tfrac{\pi}{4} - z)\ tang.\,(\tfrac{\pi}{4} + z)\ tang.\,(\tfrac{2\pi}{4} - z)$$

$$1 = tang.\,z.\,tang.\,(\tfrac{\pi}{6} - z)\ tang.\,(\tfrac{\pi}{6} + z)\ tang.\,(\tfrac{2\pi}{6} - z) \times tang.\,(\tfrac{2\pi}{6} + z)\ tang.\,(\tfrac{3\pi}{6} - z)$$

&c.

Harum vero æquationum ratio statim sponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter est alterius complementum ad rectum. Hujusmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet esse æquale.

258. Quoniam Sinus & Cosinus Angulorum progressionem arithmeticam constituentium Seriem recurrentem præbent, per Caput præcedens summa hujusmodi Sinuum & Cosinuum quotcunque exhiberi poterit. Sint Anguli in arithmetica progressione

$$a,\ a + b,\ a + 2b,\ a + 3b,\ a + 4b,\ a + 5b,\ \&c.$$

& quæratur primo summa Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

 $s = \sin. a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \&c.$ & quia hæc Series est recurrens, cujus scala relationis est $2 \cos. b, -1$, orietur hæc Series ex evolutione fractionis, cujus denominator est $1 - 2z. \cos. b + zz$, posito $z = 1$. Ipsa vero fractio erit $= \frac{\sin. a + z(\sin.(a+b) - 2 \sin. a. \cos. b)}{1 - 2z. \cos. b + zz}$, quare, facto $z = 1$, erit $s = \frac{\sin. a + \sin.(a+b) - 2 \sin. a. \cos. b}{2 - 2 \cos. b} = \frac{\sin a - \sin.(a-b)}{2(1 - \cos. b)}$, ob $2 \sin. a. \cos. b = \sin.(a+b) + \sin.(a-b)$. Cum autem sit $\sin. f - \sin. g = 2 \cos. \frac{f+g}{2} . \sin. \frac{f-g}{2}$, erit $\sin. a - \sin.(a-b) = 2 \cos.(a - \frac{1}{2} b) \sin. \frac{1}{2} b$: & $1 - \cos. b = 2(\sin. \frac{1}{2} b)^2$, unde erit $s = \frac{\cos.(a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b}$.

259. Hinc itaque summa quotcunque Sinuum, quorum Arcus in arithmetica progressione incedunt, assignari poterit; quæratur nempe summa hujus progressionis

$\sin. a + \sin.(a+b) + \sin.(a+2b) + \sin.(a+3b) + \ldots\ldots + \sin.(a+nb).$

Quia summa hujus progressionis in infinitum continuatæ est $\frac{\cos.(a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b}$, considerentur termini ultimum sequentes in infinitum hi

$\sin.(a+(n+1)b) + \sin.(a+(n+2)b) + \sin.(a+(n+3)b) + \&c.$

quia horum Sinuum summa est $= \frac{\cos.(a+(n+\frac{1}{2})b)}{2 \sin. \frac{1}{2} b}$, si hæc a priori subtrahatur, remanebit summa quæsita. Scilicet, si fuerit

$s = \sin. a + \sin.(a+b) + \sin.(a+2b) + \ldots\ldots\ldots + \sin.(a+nb)$, erit $s = \frac{\cos.(a - \frac{1}{2} b) - \cos.(a + (n + \frac{1}{2}) b)}{2 \sin. \frac{1}{2} b} = \frac{\sin(a + \frac{1}{2} nb) \sin. \frac{1}{2}(n+1) b}{\sin. \frac{1}{2} b}$.

260. Pari modo, si consideretur summa Cosinuum, atque ponatur

$$s = \cos. a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \&c.$$

in infinitum, erit $s = \frac{\cos. a + z(\cos.(a+b) - 2\cos. a.\cos. b)}{1 - 2z.\cos. b + zz}$, posito $z=1$. Quare, ob $2\cos. a.\cos. b = \cos.(a-b) + \cos.(a+b)$, fiet $s = \frac{\cos. a - \cos.(a-b)}{2(1-\cos. b)}$. At est $\cos. f - \cos. g = 2\sin.\frac{f+g}{2} \times \sin.\frac{g-f}{2}$; unde erit $\cos. a - \cos.(a-b) = -2\sin.(a-\frac{1}{2}b) \times \sin.\frac{1}{2}b$, & ob $1 - \cos. b = 2(\sin.\frac{1}{2}b)^2$, erit $s = -\frac{\sin.(a-\frac{1}{2}b)}{2\sin.\frac{1}{2}b}$. Quare, cum simili modo sit hujus Seriei

$$\cos.(a+(n+1)b) + \cos.(a+(n+2)b) + \cos.(a+(n+3)b) + \&c.$$

summa $= -\frac{\sin.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b}$, si hæc ab illa subtrahatur, relinquetur summa hujus Seriei

$$s = \cos. a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \ldots\ldots + \cos.(a+nb):$$

eritque $s = -\frac{\sin.(a-\frac{1}{2}b) + \sin.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b} = \frac{\cos.(a+\frac{1}{2}nb)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}$.

261. Plurimæ aliæ quæstiones circa Sinus & Tangentes ex principiis allatis resolvi possent; cujusmodi sunt, si quadrata, altioresve Potestates Sinuum, Tangentiumve summari deberent, verum quia hæc ex reliquis æquationum superiorum coëfficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque Sinuum Cosinuumque Potestatem per singulos Sinus Cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

262. Ad hoc expediendum juvabit ex præcedentibus hæc Lemmata depromſiſſe

$$2\sin. a. \sin. z = \cos. (a - z) - \cos. (a + z)$$
$$2\cos. a. \sin. z = \sin. (a + z) - \sin. (a - z)$$
$$2\sin. a. \cos. z = \sin. (a + z) + \sin. (a - z)$$
$$2\cos. a. \cos. z = \cos. (a - z) + \cos. (a + z).$$

Hinc igitur primum Poteſtates Sinuum reperiuntur

$$\sin. z = \sin. z$$
$$2 (\sin. z)^2 = 1 - \cos. 2z$$
$$4 (\sin. z)^3 = 3\sin. z - \sin. 3z$$
$$8 (\sin. z)^4 = 3 - 4\cos. 2z + \cos. 4z$$
$$16 (\sin. z)^5 = 10\sin. z - 5\sin. 3z + \sin. 5z$$
$$32 (\sin. z)^6 = 10 - 15\cos. 2z + 6\cos. 4z - \cos. 6z$$
$$64 (\sin. z)^7 = 35\sin. z - 21\sin. 3z + 7\sin. 5z - \sin. 7z$$
$$128 (\sin. z)^8 = 35 - 56\cos. 2z + 28\cos. 4z - 8\cos. 6z + \cos. 8z$$
$$256 (\sin. z)^9 = 126\sin. z - 84\sin. 3z + 36\sin. 5z - 9\sin. 7z + \sin. 9z$$

&c.

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, niſi quod numerus abſolutus in Poteſtatibus paribus ſemiſſis tantum ſit ejus, quem unciæ præbent.

263. Pari modo Poteſtates Coſinuum definientur

$$\cos. z = \cos. z$$
$$2 (\cos. z)^2 = 1 + \cos. 2z$$
$$4 (\cos. z)^3 = 3\cos. z + \cos. 3z$$
$$8 (\cos. z)^4 = 3 + 4\cos. 2z + \cos. 4z$$
$$16 (\cos. z)^5 = 10\cos. z + 5\cos. 3z + \cos. 5z$$
$$32 (\cos. z)^6 = 10 + 15\cos. 2z + 6\cos. 4z + \cos. 6z$$
$$64 (\cos. z)^7 = 35\cos. z + 21\cos. 3z + 7\cos. 5z + \cos. 7z$$

&c.

Hic ratione legis progreſſionis eadem ſunt monenda quæ circa Sinus notavimus.

CAPUT XV.

De Seriebus ex evolutione Factorum ortis.

264. Sit propoſitum productum ex Factoribus, numero ſive finitis ſive infinitis, conſtans hujuſmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\epsilon z)(1+\zeta z)\text{\&c.},$$

quod, ſi per multiplicationem actualem evolvatur, det

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{\&c.},$$

atque manifeſtum eſt coëfficientes A, B, C, D, E, &c., ita formari ex numeris α, β, γ, δ, ϵ, ζ, &c., ut ſit

$A = \alpha + \beta + \gamma + \delta + \epsilon + \zeta +$ &c. $=$ ſummæ ſingulorum
$B =$ ſummæ Factorum ex binis diverſis
$C =$ ſummæ Factorum ex ternis diverſis
$D =$ ſummæ Factorum ex quaternis diverſis
$E =$ ſummæ Factorum ex quinis diverſis
&c.

donec perveniatur ad productum ex omnibus.

265. Quod ſi ergo ponatur $z = 1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)\text{\&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ϵ, &c., vel ſumendis ſingulis, vel duobus pluribuſve diverſis in ſe multiplicandis, naſcuntur. Atque ſi idem numerus duobus pluribuſve modis reſultare queat, etiam idem bis plurieſve in hac numerorum Serie occurret.

266. Si ponatur $z = -1$, productum hoc

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)\ \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ε, ζ, &c. vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis, vel ternis, vel quinis, vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis, vel quaternis, vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro α, β, γ, δ, &c., numeri primi omnes 2, 3, 5, 7, 11, 13, &c., atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13)\ \&c. = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipsorum, vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+\&c.,$$

in qua Serie omnes occurrunt numeri naturales, exceptis Potestatibus, iisque qui per quamvis Potestatem sunt divisibiles. Desunt scilicet numeri 4, 8, 9, 12, 16, 18, &c., quoniam sunt vel Potestates, ut 4, 8, 9, 16, &c., vel per Potestates divisibiles ut 12, 18, &c.

268. Simili modo res se habebit, si pro α, β, γ, δ, &c., Potestates quæcunque numerorum primorum substituantur. Scilicet si ponamus

$$P = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\ \&c..$$

Erit enim multiplicatione instituta:

$$P = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\&c.$$

in quibus fractionibus omnes occurrunt numeri præter illos qui vel ipsi sunt Potestates, vel per Potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

269. Si numeri $\alpha, \beta, \gamma, \delta$, &c., negative capiantur, ut ante (266) fecimus, atque ponatur

$$P = (1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \text{ \&c., erit}$$

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \text{\&c.},$$

ubi iterum, ut ante, omnes occurrunt numeri præter Potestates ac divisibiles per Potestates. Verum ipsi numeri primi, & qui ex ternis, quinis, numerove imparibus constant, signum habent præfixum —, qui autem ex binis, vel quaternis, vel senis, vel numero paribus formantur, signum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia est $30 = 2 \cdot 3 \cdot 5$, neque adeo Potestatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ signum —, quia 30 est productum ex tribus numeris primis.

270. Consideremus jam hanc expressionem

$$\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \epsilon z) \text{ \&c.}}$$

quæ per divisionem actualem evoluta præbeat hanc Seriem :

Lib. I. $1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$
atque manifeſtum eſt coëfficientes A, B, C, D, E, &c., ſequenti modo ex numeris, $\alpha, \beta, \gamma, \delta, \epsilon$, &c., componi, ut ſit

A = ſummæ ſingulorum
B = ſummæ Factorum ex binis
C = ſummæ Factorum ex ternis
D = ſummæ Factorum ex quaternis
&c.
} non excluſis Factoribus iiſdem.

271. Poſito ergo $z = 1$, iſta expreſſio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\epsilon)\ \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c., vel ſumendis ſingulis, vel duobus pluribuſve in ſe multiplicandis, oriuntur, non excluſis æqualibus. Hoc ergo differt iſta numerorum Series ab illa, quæ (§. 265) prodiit, quod ibi Factores tantum diverſi ſumi debebant, hic autem idem Factor bis pluriesve occurrere poſſit. Hic ſcilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta$, &c., provenire poſſunt.

272. Hanc ob rem Series ſemper ex terminorum numero infinito conſtat, ſive Factorum numerus fuerit infinitus, ſive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adſunt, qui ex binario ſolo per multiplicationem oriuntur, ſeu omnes binarii Poteſtates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; seu qui alios Divisores præter 2 & 3 non habent. CAP. XV.

273. Si igitur pro α, β, γ, δ, &c., unitas per singulos omnes numeros primos divisa scribatur, ac ponatur

$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\text{\&c.}},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{\&c.},$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi, vel ex primis per multiplicationem oriundi, manifestum est, hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum Potestates quæcunque accipiantur: si enim ponatur

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\text{\&c.}},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c.},$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod si autem in Factoribus ubique signum + statuatur, ut sit

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\text{\&c.}},$$

erit

LIB. I. $P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$

ubi numeri primi habent ſignum $-$; qui ſunt producti ex duobus primis, ſive iiſdem ſive diverſis, ſignum habent $+$; & generatim, quorum numerorum numerus Factorum primorum eſt par, ſignum habent $+$, qui autem ex Factoribus primis numero imparibus conſtant, habent ſignum $-$. Sic terminus $\frac{1}{240^n}$, ob $240 = 2.2.2.2.3.5$, habebit ſignum $+$, cujus legis ratio percipitur ex §. 270, ſi ponatur $z = -1$.

275. Si hæc cum ſuperioribus conferantur, naſcentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

&

$$Q = (1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.,$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n}\ \&c.$$

(269), atque manifeſtum eſt fore $P\,Q = 1$.

276. Sin autem ponatur

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.},$$

&

$$Q = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})\ \&c.,$$ CAP. XV.

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \&c.$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

ſimilique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei ſumma, ſimul alterius innoteſcet.

277. Viciſſim porro ex cognitis ſummis harum Serierum, aſſignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \&c.,$$

eritque

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\ \&c.}$$

$$N = \frac{1}{(1 - \frac{1}{2^{2n}})(1 - \frac{1}{3^{2n}})(1 - \frac{1}{5^{2n}})(1 - \frac{1}{7^{2n}})(1 - \frac{1}{11^{2n}})\ \&c.}$$

Hinc per diviſionem naſcitur

$$\frac{M}{N} = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})\ \&c.$$

denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1} \cdot \frac{7^n+1}{7^n-1} \cdot \frac{11^n+1}{11^n-1} \cdot \&c.$$

Ex cognitis ergo M & N, præter valores horum productorum, ſummæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ex quarum combinatione multæ aliæ deduci poſſunt.

Exemplum I.

Sit $n = 1$, &, quoniam ſupra demonſtravimus eſſe, $l\,\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$, erit, poſito $x = 1$, $l\,\frac{1}{1-1} = l\,\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ At Logarithmus numeri infinite magni ∞ ipſe eſt infinite magnus, ex quo erit

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \; \&c..$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\&c.},$$

unde fit

$$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \&c.,$$

&

$$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \cdot \&c..$$

Deinde per summationem Serierum supra traditam erit

$$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6},$$ hinc obtinentur istæ summæ Serierum

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \; \&c.$$

Denique pro Factoribus orietur

LIB. I. $\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot$ &c.,

ſeu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \text{\&c.}$$

&, ob $\frac{M}{N} = \infty$ ſeu $\frac{N}{M} = 0$, habebitur

$$\infty = \frac{3}{2} \cdot \frac{4}{4} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \text{\&c.},$$

ſeu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \text{\&c.},$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \text{\&c.},$$

ſeu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \text{\&c.},$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, ſummæ autem ex numeratoribus & denominatoribus cujuſque fractionis conſtanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

EXEMPLUM II.

Sit $n = 2$, eritque ex ſuperioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{\&c.} = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \text{\&c.} = \frac{\pi^4}{90}$$

Hinc primo iſtæ Series ſummantur

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{\&c.}$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{\&c.}$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \text{\&c.}$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} \text{ \&c.},$$

Deinde valores sequentium productorum innotescunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \text{\&c.}$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{\&c.}$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \text{\&c.},$$

seu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \text{\&c.},$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \text{\&c.},$$

sive

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.},$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.}.$$

In his fractionibus numeratores unitate superant denominatores, simul vero sumti præbent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2, &c.

EXEMPLUM III.

Quia ex ſuperioribus valores ipſius M tantum ſi n ſit numerus par, aſſignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ ſequentes Series ſummantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} \&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} \&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} \&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} \&c.,$$

Deinde etiam valores ſequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{\pi^8}{9{,}50} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5}{5^8-1} \cdot \frac{7}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \&c.$$

&

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7-1} \cdot \frac{11^4+1}{11^4-1} \cdot \&c.$$

ſeu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c. CAP. XV.

278. Quoniam hic ſummam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum ſit

$$M = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

erit

$$lM = -l(1-\frac{1}{2^n}) - l(1-\frac{3}{3^n}) - l(1-\frac{1}{5^n}) - l(1-\frac{1}{7^n}) - \&c..$$

Hinc, ſumendis Logarithmis hyperbolicis, erit

$$\begin{aligned} lM = &+ 1(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.) \\ &+ \frac{1}{2}(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.) \\ &+ \frac{1}{3}(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.) \\ &+ \frac{1}{4}(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.) \\ &\&c. \end{aligned}$$

Quod ſi inſuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \&c.,$$

ut sit

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\ \&c.},$$

fiet, Logarithmis hyperbolicis sumendis,

$$\begin{aligned} lN = &+ 1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c.\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c.\right) \\ &\&c. \end{aligned}$$

Ex his conjunctis fiet $\quad lM - \frac{1}{2}lN =$

$$\begin{aligned} &+ 1\left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \&c.\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.\right) \\ &+ \frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c.\right) \\ &+ \frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c.\right) \\ &\&c. \end{aligned}$$

279. Si $n = 1$, erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ $= l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l\infty - \frac{1}{2}\, l\frac{\pi\pi}{6} =$

$$
\begin{aligned}
&+ 1\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \&c.\right)\\
&+ \tfrac{1}{3}\left(\tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{5^3} + \tfrac{1}{7^3} + \tfrac{1}{11^3} + \&c.\right)\\
&+ \tfrac{1}{5}\left(\tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{5^5} + \tfrac{1}{7^5} + \tfrac{1}{11^5} + \&c.\right)\\
&+ \tfrac{1}{7}\left(\tfrac{1}{2^7} + \tfrac{1}{3^7} + \tfrac{1}{5^7} + \tfrac{1}{7^7} + \tfrac{1}{11^7} + \&c.\right)\\
&\qquad\qquad \&c.
\end{aligned}
$$

Verum hæ Series, præter primam, non ſolum ſummas habent finitas, ſed etiam cunctæ ſimul ſumtæ ſummam efficiunt finitam, eamque ſatis parvam: unde neceſſe eſt ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, ſumma ſit infinite magna, quantitate ſcilicet ſatis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\pi^4}{90}$: unde fit

$$
\begin{aligned}
2\,l\pi - l6 = {} & 1\left(\tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{11^2} + \&c.\right)\\
&+ \tfrac{1}{2}\left(\tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \tfrac{1}{11^4} + \&c.\right)\\
&+ \tfrac{1}{3}\left(\tfrac{1}{2^6} + \tfrac{1}{3^6} + \tfrac{1}{5^6} + \tfrac{1}{7^6} + \tfrac{1}{11^6} + \&c.\right)\\
&\qquad\qquad \&c.
\end{aligned}
$$

$$4\,l\pi - l\,90 = +\ 1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$
&c.

$$lM - \frac{1}{2}lN = \frac{1}{2}\,l\,\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$
&c.

281. Quanquam lex, qua numeri primi progrediuntur, non conſtat, tamen harum Seriesum altiorum Poteſtatum ſummæ non difficulter proxime aſſignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$
&
$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.,$$
erit
$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$
& ob
$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$
erit
$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.,$$
ſeu

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam ſummam M, valor ipſius S commode invenitur, ſi quidem n fuerit numerus mediocriter magnus.

282. Inventis autem ſummis altiorum Poteſtatum, etiam ſummæ Poteſtatum minorum ex formulis inventis exhiberi poſſunt. Atque hac methodo ſequentes prodierunt ſummæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

ſi ſit	erit ſumma Seriei
$n = 2$;	0, 452247420041222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

LIB. I.

$n = 20$;	0, 000000953961123
$n = 22$;	0, 000000238450446
$n = 24$;	0, 000000059608184
$n = 26$;	0, 000000014901555
$n = 28$;	0, 000000003725333
$n = 30$;	0, 000000000931323
$n = 32$;	0, 000000000232830
$n = 34$;	0, 000000000058207
$n = 36$;	0, 000000000014551

reliquæ ſummæ parium Poteſtatum in ratione quadrupla decreſcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c., in productum infinitum converſio etiam directe inſtitui poteſt hoc modo: ſit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c.},$$ ſubtrahe

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{\&c.},$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

$= B$: ſic ſublati ſunt omnes termini per 2 diviſibiles,

ſubtr. $\frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} +$ &c.,

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{\&c.} = C:$$

ſic inſuper ſublati ſunt omnes termini per 3 diviſibiles,

$$\text{subtr.}\ \frac{1}{5^n}C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$$

erit

$$\left(1 - \frac{1}{5^n}\right)C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

ſic ſublati etiam ſunt omnes termini per 5 diviſibiles. Pari modo tolluntur termini diviſibiles per 7, 11, reliquoſque numeros primos; manifeſtum autem eſt ſublatis omnibus terminis, qui per numeros primos diviſibiles ſint, ſolam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus reſtitutis tandem orietur

$$A\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\&c. = 1,$$

unde Seriei propoſitæ ſumma erit =

$$A = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\&c.},$$

ſeu

$$A = \frac{2^n}{2^n-1}\cdot\frac{3^n}{3^n-1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n-1}\cdot\frac{11^n}{11^n-1}\cdot\&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum ſummas ſupra invenimus, in producta infinita convertendas. Invenimus autem ſupra (175) ſummas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

ſi n fuerit numerus impar, ſumma enim eſt $= N\pi^n$ & valores ipſius N loco citato dedimus. Notandum autem eſt

LIB. I. cum hic tantum numeri impares occurrunt, eos qui sint formæ $4m+1$ habere signum $+$, reliquos formæ $4m-1$ signum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \&c.$$

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{3^n}\right) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} \&c., = B$$

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} \&c.,$$ subtrahatur, erit

$$\left(1 - \frac{1}{5^n}\right) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = C,$$

ubi jam numeri per 3 & 5 divisibiles desunt,

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = D,$$

sic etiam numeri per 7 divisibiles sunt sublati,

$$\frac{1}{11^n} D = \frac{1}{11^n} - \frac{1}{121^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{11^n}\right) D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = E,$$

sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus, tandem prodibit CAP. XV.

$$A(1+\frac{1}{3^n})(1-\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})(1-\frac{1}{13^n})\ \&c. = 1,$$

seu

$$A = \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \times \frac{17^n}{17^n-1} \cdot \&c.,$$

ubi in numeratoribus occurrunt Potestates omnium numerorum primorum, quæ in denominatoribus insunt unitate sive auctæ sive minutæ, prout numeri primi fuerint formæ $4m-1$, vel $4m+1$.

285. Posito ergo $n=1$, ob $A=\frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.,$$

supra autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2.4} \cdot \frac{5^2}{4.6} \cdot \frac{7^2}{6.8} \cdot \frac{11^2}{10.12} \cdot \frac{13^2}{12.14} \cdot \frac{17^2}{16.18} \cdot \frac{19^2}{18.20} \cdot \&c..$$

Dividatur secunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c,$$

seu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares, unitate differentes a nume-

LIB. I. ratoribus. Quod si hæc denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{21}{22} \cdot \&c.,$$

seu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes dispescendo, & partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hæ expressiones cum *Wallisiana* comparentur

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11}\ \&c.,$$

seu

$$\frac{4}{\pi} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{11.\ 11}{10.\ 12} \cdot \&c.,$$

cum sit

$$\frac{\pi\pi}{8} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{11.\ 11}{10.\ 12} \cdot \frac{13.\ 13}{12.\ 14} \cdot \&c.,$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9.\ 9}{8.\ 10} \cdot \frac{15.\ 15}{14.\ 16} \cdot \frac{21.\ 21}{20.\ 22} \cdot \frac{25.\ 25}{24.\ 26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$, erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1} \cdot \frac{5^3}{5^3-1} \cdot \frac{7^3}{7^3+1} \cdot \frac{11^3}{11^3+1} \cdot \frac{13^3}{13^3-1} \cdot \frac{17^3}{17^3-1} \cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot$$
&c., seu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \text{\&c.},$$

quæ per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot$$
&c.,

hæc vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot$$
&c., seu

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \text{\&c.},$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes dispescendo, ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novæ Series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cnm enim sit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \text{\&c.},$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\text{\&c.}},$$

unde per evolutionem hæc Series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \text{\&c.},$$

 ubi ratio signorum ita est comparata, ut binarius habeat —, numeri primi formæ $4m - 1$ signum — & numeri primi formæ $4m + 1$ signum +; numeri autem compositi ea habent signa, quæ ipsis ratione multiplicationis ex primis conveniunt. Sic. patebit signum fractionis $\frac{1}{60}$, ob $60 =$
— — — +
2 . 2 . 3 . 5, quod erit —. Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\,\&c.},$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\,\&c.,$$

ubi binarius habet signum +, numeri primi formæ $4m - 1$ signum —, numeri primi formæ $4m + 1$ signum + & numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit, secundum regulas multiplicationis.

289. Cum deinde sit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}\,\&c.,$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formæ $4m - 1$ signum habeant +, numeri primi formæ $4m + 1$ signum —, unde simul numerorum compositorum signa definiuntur. Binæ porro

Series hinc formari possunt, ubi omnes numeri occurrunt; erit scilicet CAP. XV.

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\,\&c.,$$

ubi binarius signum habet $+$, numeri primi formæ $4m-1$ signum $+$, numeri vero primi formæ $4m+1$ signum $-$. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\,\&c.,$$

ubi binarius habet signum $-$, numeri primi formæ $4m-1$ signum $+$ & numeri primi formæ $4m+1$ signum $-$.

290. Possunt hinc etiam innumerabiles aliæ signorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

summa assignari queat. Cum scilicet sit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\,\&c.}.$$

Multiplicetur hæc expreſſio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.},$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.,$$

ubi binarius ſignum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m-1$ ſignum $-$; at numeri primi formæ $4m+1$ ſignum $+$; unde pro numeris compoſitis ratio ſignorum intelligitur. Simili modo, cum ſit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})(1+\frac{1}{17})\ \&c.},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet ſignum $+$, numeri primi formæ $4m-1$ ſignum $+$ & numeri primi formæ $4m+1$, præter quinarium, ſignum $-$.

291. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa sit $= 0$. Cum enim sit C A P. XV.

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\,\&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\,\&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{\&c.},$$

ut sit

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\text{\&c.}},$$

fiet, Logarithmis hyperbolicis sumendis,

$$\begin{aligned} lN = &+ 1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{\&c.}\right)\\ &+ \frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{\&c.}\right)\\ &+ \frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \text{\&c.}\right)\\ &+ \frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \text{\&c.}\right)\\ &\text{\&c.} \end{aligned}$$

Ex his conjunctis fiet $lM - \frac{1}{2}lN =$

$$\begin{aligned} &+ 1\left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \text{\&c.}\right)\\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{\&c.}\right)\\ &+ \frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \text{\&c.}\right)\\ &+ \frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \text{\&c.}\right)\\ &\text{\&c.} \end{aligned}$$

279. Si $n = 1$, erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$
$= l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l\infty - \frac{1}{2}\, l\frac{\pi\pi}{6} =$

$$+ 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \&c.\right)$$
$$+ \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \&c.\right)$$
&c.

Verum hæ Series, præter primam, non ſolum ſummas habent finitas, ſed etiam cunctæ ſimul ſumtæ ſummam efficiunt finitam, eamque ſatis parvam: unde neceſſe eſt ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, ſumma ſit infinite magna, quantitate ſcilicet ſatis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\pi^4}{90}$: unde fit

$$2\,l\pi - l6 = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
&c.

$$4\,l\pi - l\,90 = +\ 1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$
$$\&c.$$

$$lM - \frac{1}{2}lN = \frac{1}{2}\,l\,\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$
$$\&c.$$

281. Quanquam lex, qua numeri primi progrediuntur, non conſtat, tamen harum Serierum altiorum Poteſtatum ſummæ non difficulter proxime aſſignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$

&

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.,$$

erit

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$

& ob

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$

erit

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.,$$

ſeu

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam fummam M, valor ipfius S commode invenitur, fi quidem n fuerit numerus mediocriter magnus.

282. Inventis autem fummis altiorum Poteftatum, etiam fummæ Poteftatum minorum ex formulis inventis exhiberi poffunt. Atque hac methodo fequentes prodierunt fummæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

fi fit	erit fumma Seriei
$n = 2$;	0, 452247420041222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

Lib. I.

$n = 20$;	0, 000000953961123
$n = 22$;	0, 000000238450446
$n = 24$;	0, 000000059608184
$n = 26$;	0, 000000014901555
$n = 28$;	0, 000000003725333
$n = 30$;	0, 000000000931323
$n = 32$;	0, 000000000232830
$n = 34$;	0, 000000000058207
$n = 36$;	0, 000000000014551

reliquæ ſummæ parium Poteſtatum in ratione quadrupla decreſcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$, in productum infinitum converſio etiam directe inſtitui poteſt hoc modo: ſit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ſubtrahe

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \&c.,$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \&c. = B:$$

ſic ſublati ſunt omnes termini per 2 diviſibiles,

$$\text{ſubtr. } \frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c. = C:$$

ſic inſuper ſublati ſunt omnes termini per 3 diviſibiles,

$$\text{fubtr. } \frac{1}{5^n} C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$$

erit

$$\left(1 - \frac{1}{5^n}\right) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

fic fublati etiam funt omnes termini per 5 divifibiles. Pari modo tolluntur termini divifibiles per 7, 11, reliquofque numeros primos; manifeftum autem eft fublatis omnibus terminis, qui per numeros primos divifibiles fint, folam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus reftitutis tandem orietur

$$A\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)\&c. = 1,$$

unde Seriei propofitæ fumma erit =

$$A = \frac{1}{\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)\&c.},$$

feu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum fummas fupra invenimus, in producta infinita convertendas. Invenimus autem fupra (175) fummas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

fi n fuerit numerus impar, fumma enim eft $= N\pi^n$ & valores ipfius N loco citato dedimus. Notandum autem eft

LIB. I. cum hic tantum numeri impares occurrunt, eos qui ſint formæ $4m+1$ habere ſignum $+$, reliquos formæ $4m-1$ ſignum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \&c.$$

$$\frac{1}{3^n}A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{3^n}\right)A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n}\ \&c., = B$$

$$\frac{1}{5^n}B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n}\ \&c.,$$ ſubtrahatur, erit

$$\left(1 - \frac{1}{5^n}\right)B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = C,$$

ubi jam numeri per 3 & 5 diviſibiles deſunt,

$$\frac{1}{7^n}C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = D,$$

ſic etiam numeri per 7 diviſibiles ſunt ſublati,

$$\frac{1}{11^n}D = \frac{1}{11^n} - \frac{1}{121^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{11^n}\right)D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = E,$$

ſic numeri per 11 diviſibiles quoque ſunt ſublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus, tandem prodibit C A P. XV.

$$A\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right) \&c. = 1,$$

seu

$$A = \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \times \frac{17^n}{17^n-1} \cdot \&c.,$$

ubi in numeratoribus occurrunt Potestates omnium numerorum primorum, quæ in denominatoribus insunt unitate sive auctæ sive minutæ, prout numeri primi fuerint formæ $4m-1$, vel $4m+1$.

285. Posito ergo $n=1$, ob $A=\frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.,$$

supra autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2.4} \cdot \frac{5^2}{4.6} \cdot \frac{7^2}{6.8} \cdot \frac{11^2}{10.12} \cdot \frac{13^2}{12.14} \cdot \frac{17^2}{16.18} \cdot \frac{19^2}{18.20} \cdot \&c..$$

Dividatur secunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c,$$

seu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares, unitate differentes a nume-

LIB. I. ratoribus. Quod si hæc denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{24}{22} \cdot \&c.,$$

seu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes dispescendo, & partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hæ expressiones cum *Wallisiana* comparentur

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11}\ \&c.,$$

seu

$$\frac{4}{\pi} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{11.\ 11}{10.\ 12} \cdot \&c.,$$

cum sit

$$\frac{\pi\pi}{8} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{11.\ 11}{10.\ 12} \cdot \frac{13.\ 13}{12.\ 14} \cdot \&c.,$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9.\ 9}{8.\ 10} \cdot \frac{15.\ 15}{14.\ 16} \cdot \frac{21.\ 21}{20.\ 22} \cdot \frac{25.\ 25}{24.\ 26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$, erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1} \cdot \frac{5^3}{5^3-1} \cdot \frac{7^3}{7^3+1} \cdot \frac{11^3}{11^3+1} \cdot \frac{13^3}{13^3-1} \cdot \frac{17^3}{17^3-1} \cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$$\frac{\pi^6}{945} = \frac{2^6}{2^6 - 1} \cdot \frac{3^6}{3^6 - 1} \cdot \frac{5^6}{5^6 - 1} \cdot \frac{7^6}{7^6 - 1} \cdot \frac{11^6}{11^6 - 1} \cdot \frac{13^6}{13^6 - 1} \cdot \text{\&c., seu}$$

$$\frac{\pi^6}{960} = \frac{3^6}{3^6 - 1} \cdot \frac{5^6}{5^6 - 1} \cdot \frac{7^6}{7^6 - 1} \cdot \frac{11^6}{11^6 - 1} \cdot \frac{13^6}{13^6 - 1} \cdot \text{\&c.},$$

quæ per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3 - 1} \cdot \frac{5^3}{5^3 + 1} \cdot \frac{7^3}{7^3 - 1} \cdot \frac{11^3}{11^3 - 1} \cdot \frac{13^3}{13^3 + 1} \cdot \frac{17^3}{17^3 + 1} \cdot \text{\&c.},$$

hæc vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3 + 1}{3^3 - 1} \cdot \frac{5^3 - 1}{5^3 + 1} \cdot \frac{7^3 + 1}{7^3 - 1} \cdot \frac{11^3 + 1}{11^3 - 1} \cdot \frac{13^3 - 1}{13^3 + 1} \cdot \frac{17^3 - 1}{17^3 + 1} \cdot \text{\&c., seu}$$

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \text{\&c.},$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes dispescendo, ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novæ Series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \text{\&c.},$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\text{\&c.}},$$

unde per evolutionem hæc Series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \text{ \&c.},$$

 ubi ratio signorum ita est comparata, ut binarius habeat —, numeri primi formæ $4m - 1$ signum — & numeri primi formæ $4m + 1$ signum +; numeri autem compositi ea habent signa, quæ ipsis ratione multiplicationis ex primis conveniunt. Sic. patebit signum fractionis $\frac{1}{60}$, ob $60 = \overset{-\;\;-\;\;-\;\;+}{2 \cdot 2 \cdot 3 \cdot 5}$, quod erit —. Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\,\&c.},$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet signum +, numeri primi formæ $4m - 1$ signum —, numeri primi formæ $4m + 1$ signum + & numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit, secundum regulas multiplicationis.

289. Cum deinde sit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}\ \&c.,$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formæ $4m - 1$ signum habeant +, numeri primi formæ $4m + 1$ signum —, unde simul numerorum compositorum signa definiuntur. Binæ porro

Series hinc formari possunt, ubi omnes numeri occurrunt, erit scilicet

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\,\&c.,$$

ubi binarius signum habet $+$, numeri primi formæ $4m - 1$ signum $+$, numeri vero primi formæ $4m + 1$ signum $-$. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\,\&c.,$$

ubi binarius habet signum $-$, numeri primi formæ $4m - 1$ signum $+$ & numeri primi formæ $4m + 1$ signum $-$.

290. Possunt hinc etiam innumerabiles aliæ signorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

summa assignari queat. Cum scilicet sit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\,\&c.}.$$

Multiplicetur hæc expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.},$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.,$$

ubi binarius signum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m - 1$ signum $-$; at numeri primi formæ $4m + 1$ signum $+$; unde pro numeris compositis ratio signorum intelligitur. Simili modo, cum sit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})(1+\frac{1}{17})\ \&c.},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet signum $+$, numeri primi formæ $4m - 1$ signum $+$ & numeri primi formæ $4m + 1$, præter quinarium, signum $-$.

291. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa sit $= 0$. Cum enim sit CAP. XV.

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\&c.},$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\&c.},$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

LIB. I. $0 = \cfrac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\&c.},$

unde oritur iſta Series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi omnes numeri primi, præter 3 & 5, habent ſignum —. In genere autem notandum eſt, quoties omnes numeri primi, exceptis tantum aliquibus, habebunt ſignum —, ſummam Seriei fore $= 0$. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habebunt ſignum +, tum ſummam Seriei fore infinite magnam.

292. Supra etiam (176) ſummam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} \&c.,$$

ſi fuerit n numerus impar: Erit ergo

$$\frac{1}{2^n} A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n}) A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \&c.$$

$$\frac{1}{5^n} B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} \&c., \text{ addatur},$$

erit

$C =$

$C = (1 + \frac{1}{5^n}) B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n}$ &c.

$\frac{1}{7^n} C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} +$ &c., subtrahatur, erit

$$D = (1 - \frac{1}{7^n}) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{\&c.}.$$

Ex his tandem fiet

$$A (1 + \frac{1}{2^n})(1 + \frac{1}{5^n})(1 - \frac{1}{7^n})(1 + \frac{1}{11^n})(1 - \frac{1}{13^n}) \text{\&c.} = 1,$$

ubi numeri primi unitate excedentes multipla senarii habent signum $-$, deficientes autem signum $+$. Eritque

$$A = \frac{2^n}{2^n + 1} \cdot \frac{5^n}{5^n + 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \text{\&c.}.$$

293. Consideremus casum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$; eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{\&c.},$$

ubi in numeratoribus post 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate discrepant, suntque omnes per 6 divisibiles. Cum jam sit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \text{\&c.},$$

erit, hac expressione per illam divisa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{\&c.},$$

ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \&c.$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

quarum hæc per illam divisa dat

$$\frac{4}{3} = \frac{6}{4} \cdot \frac{6}{8} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{18}{16} \cdot \frac{18}{20} \cdot \frac{24}{22} \cdot \&c.,$$

seu

$$\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{12}{11} \cdot \&c.,$$

ubi singulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, singulos numeros primos in duas partes unitate differentes dispescendo, & partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra vidimus esse

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \&c.,$$

seu

$$\frac{\pi}{3} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \&c.,$$

si superiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \&c.$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \&c.,$$

In priori expressione fractiones formantur ex numeris primis formæ $12m + 6 \pm 1$, in posteriore ex numeris primis formæ $12m \pm 1$, singulos in duas partes unitate discrepantes dispescendo, & partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc Seriem supra inventam (179), quæ ita progreditur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c. = A\text{, erit}$$

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \&c.\text{: subtrahatur}$$

$$\left(1 - \frac{1}{3}\right)A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \&c. = B$$

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \&c.\text{: addatur, erit}$$

$$\left(1 + \frac{1}{5}\right)B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17}\ \&c. = C:$$

sicque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 + \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right)\ \&c. = 1,$$

ubi signa ita se habent, ut numerorum primorum formæ $8m+1$, vel $8m+3$, signa sint $-$; numerorum primorum vero formæ $8m+5$, vel $8m+7$, signa sint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \&c.,$$

ubi omnes denominatores vel divisibiles sunt per 8, vel tantum sunt numeri impariter pares. Cum igitur sit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

&

LIB. I. $\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot$ &c.,

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \text{\&c.},$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \text{\&c.},$$

quæ fractiones formantur ex numeris primis, singulos in duas partes unitate discrepantes dispescendo, & partes pares (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquæ Series, quas supra pro expressione arcuum circularium invenimus (179 *& seqq.*) in Factores transformari possunt, qui ex numeris primis constituantur. Sicque multæ aliæ insignes proprietates tam hujusmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.

CAPUT XVI.

De Partitione numerorum.

297. PROPOSITA sit ista expressio
$(1+x^\alpha z)(1+x^\beta z)(1+x^\gamma z)(1+x^\delta z)(1+x^\epsilon z)$ &c., quæ cujusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

atque manifestum est P fore summam Potestatum
$x^\alpha + x^\beta + x^\gamma + x^\delta + x^\epsilon +$ &c.. Deinde Q est summa factorum ex binis Potestatibus diversis, seu Q erit aggregatum plurium Potestatum ipsius x, quarum Exponentes sunt summæ duorum terminorum diversorum hujus Seriei

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \&c.$$

Simili modo R erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ trium terminorum diversorum. Atque S erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ quatuor terminorum diversorum ejusdem Seriei, $\alpha, \beta, \gamma, \delta, \epsilon$, &c., & ita porro.

298. Singulæ hæ Potestates ipsius x, quæ in valoribus litterarum P, Q, R, S, &c., insunt, unitatem pro coëfficiente habebunt, si quidem earum Exponentes unico modo ex $\alpha, \beta, \gamma, \delta$, &c., formari queant: sin autem ejusdem Potestatis Exponens pluribus modis possit esse summa duorum,

LIB. I. trium, pluriumve terminorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon$, &c., tum etiam Potestas illa coëfficientem habebit, qui unitatem toties in se complectatur. Sic, si in valore ipsius Q reperiatur Nx^n, indicio hoc erit numerum n esse N diversis modis summam duorum terminorum diversorum Seriei α, β, γ, &c. Atque si in evolutione Factorum propositorum occurrat terminus $Nx^n z^m$, ejus coëfficiens N indicabit quot variis modis numerus n possit esse summa m terminorum diversorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c.

299. Quod si ergo productum propositum

$$(1 + x^\alpha z)(1 + x^\beta z)(1 + x^\gamma z)(1 + x^\delta z) \text{ \&c.},$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c., quot quis voluerit. Scilicet, si quæratur quot variis modis numerus n possit esse summa m terminorum illius Seriei diversorum, in expressione evoluta quæri debet terminus $x^n z^m$, ejusque coëfficiens indicabit numerum quæsitum.

300. Quo hæc fiant planiora, sit propositum hoc productum ex Factoribus constans infinitis

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \text{ \&c.},$$

quod per multiplicationem actualem evolutum dat

$$1 + z(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \&c.)$$
$$+ z^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \&c.)$$
$$+ z^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \&c.)$$
$$+ z^4(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \&c.)$$
$$+ z^5(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \&c.)$$
$$+ z^6(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \&c.)$$
$$+ z^7(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \&c.)$$
$$+ z^8(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \&c.)$$
&c.

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus ex dato terminorum diversorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c. numero oriri queat. Sic, si quæratur quot variis modis numerus 35 possit esse summa septem terminorum diversorum Seriei 1, 2, 3, 4, 5, 6, 7, &c., quæratur in Serie z^7 multiplicante Potestatem x^{35}, ejusque coëfficiens 15 indicabit numerum propositum 35 quindecim variis modis esse summam septem terminorum Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c.

301. Quod si autem ponatur $z = 1$, & similes Potestates ipsius x in unam summam conjiciantur, seu, quod eodem redit, si evolvatur hæc expressio infinita

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)\&c.,$$

quo facto orietur hæc Series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens Potestatis ipsius x conjunctæ ex terminis diversis Seriei 1, 2, 3, 4, 5, 6, 7, &c., per additionem emergere possit. Sic apparet numerum 8 sex modis per additionem diversorum numerorum produci, qui sunt

$8 = 8$	$8 = 5 + 3$
$8 = 7 + 1$	$8 = 5 + 2 + 1$
$8 = 6 + 2$	$8 = 4 + 3 + 1$

LIB. I. ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si Factores illos in denominatorem transponamus. Sit igitur proposita hæc expressio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\epsilon}z)\,\&c.},$$

quæ per divisionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c..$$

Atque manifestum est fore P aggregatum Potestatum ipsius x, quarum Exponentes contineantur in hac Serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c..$$

Deinde Q erit aggregatum Potestatum ipsius x, quarum Exponentes sint summæ duorum terminorum hujus Seriei, sive eorundem sive diversorum. Tum erit R summa Potestatum ipsius x, quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S summa Potestatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expressio per singulos terminos explicetur, & termini similes conjunctim exprimantur, intelligetur quot variis modis propositus numerus n per additionem m terminorum, sive diversorum sive non diversorum, Seriei α, β, γ, δ, ϵ, ζ, &c., produci queat. Quæratur scilicet in expressione evoluta terminus $x^n z^m$, ejusque coëfficiens, qui sit N, ita ut totus terminus sit $= Nx^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie α, β, γ, δ, ϵ, &c., contentorum produci queat. Hoc igitur pacto quæstio priori, quam ante sumus contemplati, similis resolvetur. CAP. XVI.

304. Accommodemus hæc ad casum inprimis notatu dignum, sitque proposita hæc expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \&c.},$$

quæ per divisionem evoluta dabit

$$\begin{aligned}
1 &+ z(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \&c.)\\
&+ z^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \&c.)\\
&+ z^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \&c.)\\
&+ z^4(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \&c.)\\
&+ z^5(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \&c.)\\
&+ z^6(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \&c.)\\
&+ z^7(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \&c.)\\
&+ z^8(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \&c.)\\
&\&c.,
\end{aligned}$$

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, si quæratur quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z = 1$, atque similes Potestates ipsius x conjunctim exprimantur, hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ \&c.}$$

evolvetur in hanc Seriem

 $1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8$ &c.,

in qua quilibet coëfficiens indicat, quot variis modis Exponens Potestatis adjunctæ per additionem produci queat ex numeris integris, sive æqualibus sive inæqualibus. Scilicet ex termino $11x^6$ cognoscitur numerum 6 undecim modis per additionem numerorum integrorum produci posse, qui sunt

$6=6$	$6=3+1+1+1$
$6=5+1$	$6=2+2+2$
$6=4+2$	$6=2+2+1+1$
$6=4+1+1$	$6=2+1+1+1+1$
$6=3+3$	$6=1+1+1+1+1+1$
$6=3+2+1$	

ubi quoque notari debet, ipsum numerum propositum, cum in Serie numerorum 1, 2, 3, 4, 5, 6, &c., proposita contineatur, unum modum præbere.

306. His in genere expositis, diligentius inquiramus in modum hanc compositionum multitudinem inveniendi. Ac primo quidem consideremus eam ex numeris integris compositionem, in qua numeri tantum diversi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem proposita hæc expressio

$$Z=(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\ \&c.,$$

quæ evoluta & secundum Potestates ipsius z digesta præbeat

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\ \&c.,$$

ubi methodus desideratur has ipsius x Functiones P, Q, R, S, T, &c., expedite inveniendi, hoc enim pacto quæstioni propositæ convenientissime satisfiet.

307. Patet autem, si loco z ponatur xz, prodire $(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)$ &c. $= \frac{Z}{1+xz}$ ergo, posito xz loco z, valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; sicque, cum sit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.;$$

multiplicetur ergo actu per $1+xz$, atque prodibit

$$\begin{aligned} Z = 1 &+ Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. \\ &+ xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \&c., \end{aligned}$$

qui valor ipsius Z cum superiori comparatus dabit

$$P = \frac{x}{1-x};\ Q = \frac{Px^2}{1-x^2};\ R = \frac{Qx^3}{1-x^3};\ S = \frac{Rx^4}{1-x^4}\ \&c.,$$

Sequentes ergo pro P, Q, R, S, &c., obtinentur valores

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

&c.

308. Sic igitur seorsim unamquamque Seriem Potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem porro est has singulas Series esse recurrentes, quia ex evolutione

Functionis fractæ ipſius x naſcuntur. Prima ſcilicet expreſſio $P = \frac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifeſtum eſt quemvis numerum ſemel in Serie numerorum integrorum contineri.

309. Expreſſio ſecunda $\frac{x^3}{(1-x)(1-xx)}$, dat hanc Seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.,$$

in qua cujuſvis termini coëfficiens indicat quot modis Exponens ipſius x in duas partes inæquales diſpertiri poſſit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas partes inæquales ſecari poſſe. Quod ſi hanc Seriem per x^3 dividamus, prodibit Series, quam præbet iſta fractio $\frac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis ſit $= Nx^n$; atque ex geneſi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem naſci queat. Cum igitur prioris Seriei terminus generalis ſit $= Nx^{n+3}$, deducitur hinc iſtud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1 & 2 *produci poteſt, totidem variis modis numerus* n + 3 *in duas partes inæquales ſecari poterit.*

310. Expreſſio tertia $\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujuſvis termini coëfficiens indicat quot variis modis Exponens Poteſtatis x adjunctæ in tres partes inæquales diſper-

tiri possit. Quod si autem hæc fractio $\frac{1}{(1-x)(1-x^2)(1-x^3)}$
evolvatur, prodibit hæc Series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis si ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci possit. Cum igitur prioris Seriei terminus generalis sit Nx^{n+6}, sequetur hinc illud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1, 2, 3, *produci potest, totidem variis modis numerus* n + 6 *in tres partes inæquales secari poterit.*

311. Expressio quarta $\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.;$$

in qua cujusvis termini coëfficiens indicabit quot variis modis Exponens Potestatis x adjunctæ in quatuor partes inæquales dispertiri possit. Quod si autem hæc expressio

$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ evolvatur, prodibit superior Series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.,$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema.

Quot variis modis numerus n *per additionem produci potest*

LIB. I. *ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* n + 10 *in quatuor partes inæquales secari poterit.*

312. Generaliter ergo, si hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, ejusque terminus generalis fuerit $= Nx^n$, coëfficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4 .... m. Quod si autem hæc expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis $= Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n + \frac{m(m+1)}{1.\ 2}$ in m partes inæquales secari possit, unde hoc habetur theorema.

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3, 4 .... m, *totidem modis numerus* $n + \frac{m(m+1)}{1.\ 2}$ *in* m *partes inæquales secari poterit.*

313. Ex posita partitione numerorum in partes inæquales; perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expressione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.}$$

Ponamus evolutione per divisionem instituta prodire

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.$$

Perſpicuum autem eſt, ſi loco z ponatur xz, prodire CAP. XVI.

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.} = (1-xz)\,Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1-xz)\,Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.;$$

Multiplicetur ergo ſuperior Series pariter per $(1-xz)$, eritque

$$(1-xz)\,Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.$$
$$- xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c..$$

Comparatione ergo inſtituta orietur

$$P = \frac{x}{1-x};\; Q = \frac{Px}{1-x^2};\; R = \frac{Qx}{1-x^3};\; S = \frac{Rx}{1-x^4}\ \&c.;$$

unde pro P, Q, R, S, &c., ſequentes valores proveniunt.

$$P = \frac{x}{1-x}$$
$$Q = \frac{x^2}{(1-x)(1-x^2)}$$
$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$
$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$
&c.

314. Expreſſiones iſtæ a ſuperioribus aliter non diſcrepant, niſi quod numeratores hic minores habeant Exponentes quam caſu præcedente. Atque hanc ob rem Series, quæ per evolutionem naſcuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300 & 304.) perſpicitur, nunc vero demum ejus ratio intelligitur. Hinc ergo omnino ſimilia theoremata conſequentur, quæ ſunt.

LIB. I. *Quot variis modis numerus* n *per additionem produci potest ex numeris* 1, 2, *totidem modis numerus* $n + 2$ *in duas partes dispertiri poterit.*

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3; *totidem modis numerus* $n + 3$ *in tres partes dispertiri poterit.*

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3, 4, *totidem modis numerus* $n + 4$ *in quatuor partes dispertiri poterit.*

Atque generaliter habebitur hoc theorema :

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3, ... m, *totidem modis numerus* $n + m$ *in* m *partes dispertiri poterit.*

315. Sive ergo quæratur quot modis datus numerus in *m* partes inæquales, five in *m* partes, æqualibus non exclusis, dispertiri possit, utraque quæstio resolvetur si cognoscatur quot modis quisque numerus per additionem produci possit ex numeris 1, 2, 3, 4 .... *m*, quemadmodum hoc patebit ex sequentibus theorematibus, quæ ex superioribus sunt derivata.

Numerus n *tot modis in* m *partes inæquales dispertiri potest, quot modis numerus* $n - \frac{m(m+1)}{2}$ *per additionem produci potest ex numeris* 1, 2, 3, 4, ... m.

Numerus n, *tot modis in* m *partes five æquales five inæquales dispertiri potest quot modis numerus* $n - m$ *per additionem produci potest ex numeris* 1, 2, 3, ... m.

Hinc porro sequuntur hæc theoremata.

Numerus n *totidem modis in* m *partes inæquales secari potest, quot modis numerus* $n - \frac{m(m-1)}{2}$ *in* m *partes, five æquales five inæquales, dispertitur,*

Numerus

Numerus n *totidem modis in* m *partes, sive inæquales sive æquales, secari potest, quot modis numerus* $n + \frac{m(m-1)}{2}$ *in* m *partes inæquales dispertiri potest.* CAP. XVI.

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci possit ex numeris $1, 2, 3, \ldots\ldots\ldots m$. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

atque Series recurrens continuari debebit usque ad terminum Nx^n, cujus coëfficiens N indicabit, quot modis numerus n per additionem produci possit ex numeris $1, 2, 3, 4, \ldots\ldots m$. At vero hic solvendi modus non parum habebit difficultatis, si numeri m & n sint modice magni; scala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit Seriem ad plures terminos continuare.

317. Hæc autem disquisitio minus erit molesta, si casus simpliciores primum expediantur, ex his enim facile erit ad casus magis compositos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

terminus generalis $= Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

ortæ terminus generalis sit Mx^n, ubi coëfficiens M indica-

LIB. I. bit quot variis modis numerus $n - m$ per additionem produci possit ex numeris $1, 2, 3, \ldots\ldots m$. Subtrahatur posterior expressio a priori, ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{m-1})}$$

atque manifestum est Seriei hinc ortæ terminum generalem futurum esse $(N-M)\,x^n$; quare coëfficiens $N-M$ indicabit quot variis modis numerus n per additionem produci possit ex numeris $1, 2, 3, \ldots\ldots (m-1)$.

318. Hinc ergo sequentem regulam nanciscimur.

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris $1, 2, 3, \ldots\ldots (m-1)$.

Sit M numerus modorum, quibus numerus $n-m$ per additionem produci potest ex numeris $1, 2, 3, \ldots\ldots m$.

Sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris $1, 2, 3, \ldots\ldots m$.

His positis, erit, ut vidimus, $L = N - M$; ideoque $N = L + M$. Quod si ergo jam invenerimus quot variis modis numeri n & $n-m$ per additionem produci queant, ille ex numeris $1, 2, 3, \ldots\ldots (m-1)$, hic vero ex numeris $1, 2, 3, \ldots\ldots m$; hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris $1, 2, 3, \ldots\ldots m$. Ope hujus theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit, hocque modo tabula hic annexa est computata, * cujus usus ita se habet.

Si quæratur quot variis modis numerus 50 in 7 partes inæquales dispertiri possit; sumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quæsitum.

* *Vide Tab. pag. 275.*

Sin autem quæratur, quot variis modis numerus 50 in 7 partes, five æquales five inæquales, difpertiri poffit, in prima columna verticali fumatur numerus $50 - 7 = 43$, cui in columna 7^{ma} refpondebit numerus quæfitus 8946. CAP. XVI.

319. Series hujus tabulæ verticales, etfi funt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus & fequentibus, quam paucis exponere operæ pretium erit. Quoniam enim ex fractione $\frac{1}{(1-x)(1-xx)}$ oritur Series $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 +$ &c., ac proinde ex fractione $\frac{x}{(1-x)(1-xx)}$ hæc $x + x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 +$ &c.. Si duæ hæ Series addantur, nafcitur ifta

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \&c.,$$

quæ per divifionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} = \frac{1}{(1-x)^2}$; unde patet Serici poftremæ terminos numericos Seriem numerorum naturalium conftituere. Hinc ex Serie tabulæ fecunda addendo binos terminos proveniet Series numerorum naturalium, pofito $x = 1$.

$$1+1+2+2+3+3+4+4+5+5+6+6+\&c.$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\&c.$$

Viciffim ergo ex Serie numerorum naturalium fuperior invenitur, fubtrahendo quemque terminum Seriei fuperioris a termino inferioris fequente.

320. Series verticalis tertia oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)}$. Cum autem fit $\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$, manifeftum eft, fi primo Seriei

LIB. I. illius terni termini addantur, tum bini hujus novæ Seriei denuo addantur, prodire debere numeros trigonales, id quod ex ſchemate ſequente apparebit.

$1+1+2+3+4+5+7+8+10+12+14+16+19$ &c.
$1+2+4+6+9+12+16+20+25+30+36+42+49$ &c.
$1+3+6+10+15+21+28+36+45+55+66+78+91$ &c..

Viciſſim autem apparet quomodo ex Serie trigonalium erui debeat Series ſuperior.

321. Simili modo, quia Series quarta oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}$, erit $\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x)}$ $= \frac{1}{(1-x)^4}$. Si in Serie quarta primum quaterni termini addantur, tum in Serie reſultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex ſequenti calculo videre licet.

$1+1+2+3+5+6+9+11+15+18+23+27+$ &c.
$1+2+4+7+11+16+23+31+41+53+67+83+$ &c.
$1+3+7+13+22+34+50+70+95+125+161+203+$ &c.
$1+4+10+20+35+56+84+120+165+220+286+364+$ &c..

Simili autem modo Series quinta deducet ad numeros pyramidales ſecundi ordinis, ſexta ad tertii ordinis & ita porro.

322. Viciſſim igitur ex numeris figuratis illæ ipſæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inſpectione calculi ſequentis ſponte elucebunt.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + &c.
1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + &c. II

---

1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + &c.
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + &c.
1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + &c. III

---

1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + &c.
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + &c.
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + &c.
1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + &c. IV

---

1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + &c.
1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + &c.
1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + &c.
1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + &c.
1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + &c. V

&c.

In his ordinibus primæ Series ſunt numeri figurati, unde ſubtrahendo quemvis terminum Seriei ſecundæ a termino primæ ſequente formatur Series ſecunda. Tum Seriei tertiæ bini termini conjunctim ſubtrahantur a termino ſequente Seriei ſecundæ, ſicque oritur Series tertia; hocque modo ſubtrahendo ulterius ſummam trium, quatuor, & ita porro terminorum a termino ſuperioris Seriei ſequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab $1 + 1 + 2$ &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes ſimiliter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter ſe fore congruentes. Prodibit autem Series, quæ oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\,\&c.}$$

quæ cum ſit recurrens, primum denominator ſpectari debet, ut

LIB. I. hinc scala relationis habeatur. Quod si autem Factores denominatoris continuo in se multiplicentur, prodibit

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \&c.,$$

quæ Series si attentius consideretur, aliæ Potestates ipsius x adesse non deprehenduntur, nisi quarum Exponentes contineantur in hac formula $\frac{3nn+n}{2}$; atque, si n sit numerus impar, Potestates erunt negativæ; affirmativæ autem si n fuerit numerus par.

324. Cum igitur scala relationis sit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c..$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\,\&c.},$$

oriunda erit hæc

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} + 792x^{21} + 1002x^{22} + 1250x^{23} + 1570x^{24}\,\&c..$$

In hac ergo Serie coëfficiens quisque indicat, quot variis modis Exponens ipsius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri potest.

$7=7$	$7=4+2+1$	$7=3+1+1+1+1$
$7=6+1$	$7=4+1+1+1$	$7=2+2+2+1$
$7=5+2$	$7=3+3+1$	$7=2+2+1+1+1$
$7=5+1+1$	$7=3+2+2$	$7=2+1+1+1+1+1$
$7=4+3$	$7=3+2+1+1$	$7=1+1+1+1+1+1+1$

325. Quod si autem hoc productum CAP. XVI.

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

evolvatur, sequens prodibit Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\&c.,$$

in qua quisque coëfficiens indicat, quot variis modis Exponens ipsius x per additionem numerorum inæqualium oriri possit. Sic numerus 9 octo variis modis per additionem ex numeris inæqualibus formari potest.

$9 = 9$	$9 = 6 + 2 + 1$
$9 = 8 + 1$	$9 = 5 + 4$
$9 = 7 + 2$	$9 = 5 + 3 + 1$
$9 = 6 + 3$	$9 = 4 + 3 + 2$

326. Ut comparationem inter has formas instituamus, sit

$$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ \&c.,$$

&

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

erit

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\ \&c.,$$

qui Factores cum omnes in P contineantur, dividatur P per PQ, erit $\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.,$

ideoque

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.},$$

quæ fractio si evolvatur, prodibit Series, in qua quisque coëfficiens indicabit, quot variis modis Exponens ipsius x, per additionem ex numeris imparibus produci possit. Cum igitur hæc expressio æqualis sit illi, quam in §. præcedente contemplati sumus, sequitur hinc istud theorema.

LIB. I. *Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inæqualibus; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive æqualibus sive inæqualibus.*

327. Cum igitur, ut ante vidimus, sit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{\&c.},$$

erit, scribendo xx loco x,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \text{\&c.}.$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{\&c.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{\&c.}}.$$

Erit ergo Series Q pariter recurrens, atque ex Serie $\frac{1}{P}$ oritur, hanc per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24}$ &c. multiplicando. Nempe, cum sit (324) $\frac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.}$, si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{\&c.},$$

fiet

$$\begin{aligned} &1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.} \\ &\qquad - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{\&c.} \\ &\qquad\qquad - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{\&c.}, \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{\&c.}$$

$= Q$. Hinc ergo, si formatio numerorum per additionem numerorum, sive æqualium sive inæqualium constet, deducetur formatio numerorum per additionem numerorum inæqualium, hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe hæc expressio

(1 + x

$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$ &c., CAP. XVI.

in qua Exponentes ipſius x in ratione dupla progrediuntur. Hæc expreſſio ſi evolvatur, reperietur quidem hæc Series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \&c.;$$

quoniam vero dubium eſſe poteſt, utrum hæc Series in infinitum hac lege geometrica progrediatur, hanc ipſam Seriem inveſtigemus. Sit igitur

$$P = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ \&c.,$$

ac ponatur Series per evolutionem oriunda

$$P = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \&c.;$$

Patet autem ſi loco x ſcribatur xx, tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\ \&c. = \frac{P}{1+x}:$$

facta ergo in Serie eadem ſubſtitutione erit

$$\frac{P}{1+x} = 1 + \alpha x^2 + \beta x^4 + \gamma x^6 + \delta x^8 + \varepsilon x^{10} + \zeta x^{12} + \&c.;$$

multiplicetur ergo per $1+x$, eritque

$$P = 1 + x + \alpha x^2 + \alpha x^3 + \beta x^4 + \beta x^5 + \gamma x^6 + \gamma x^7 + \delta x^8 + \delta x^9 + \&c.,$$

qui valor ipſius P ſi cum ſuperiori comparetur, habebitur

$$\alpha = 1;\ \beta = \alpha;\ \gamma = \alpha;\ \delta = \beta;\ \varepsilon = \beta;\ \zeta = \gamma;\ \eta = \gamma;\ \&c.,$$

erunt ergo omnes coëfficientes $= 1$, ideoque productum propoſitum P evolutum dabit Seriem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c..$$

329. Cum igitur hic omnes ipſius x Poteſtates, ſingulæque ſemel occurrant, ex forma producti $(1+x)(1+x^2)(1+x^4)$&c., ſequitur, omnem numerum integrum ex terminis progreſſionis

geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diversis per additionem formari posse, hocque unico modo. Nota est hæc proprietas in praxi ponderandi, si enim habeantur pondera, 1, 2, 4, 8, 16, 32, &c., librarum; his solis ponderibus omnia onera ponderari poterunt, nisi partes libræ requirant. Sic his decem ponderibus, nempe 1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔, omnia pondera usque ad 1024 ℔, librari possunt, & si unum pondus 1024 ℔, addatur omnibus oneribus usque ad 2048 ℔, ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quæ scilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c., librarum pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent. Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet.

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$

&c.

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$(x^{-1}+1+x^{1})(x^{-3}+1+x^{3})(x^{-9}+1+x^{9})(x^{-27}+1+x^{27})\,\&c. = P,$$

quod evolutum alias non dabit Potestates ipsius x, nisi quarum Exponentes formari possint ex numeris 1, 3, 9, 27, 81, &c.,

A B U L A

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VI	VII	VIII	IX	X	XI
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
5	5	5	5	5	5
7	7	7	7	7	7
11	11	11	11	11	11
14	15	15	15	15	15
20	21	22	22	22	22
26	28	29	30	30	30
35	38	40	41	42	42
44	49	52	54	55	56
58	65	70	73	75	76
71	82	89	94	97	99
90	105	116	123	125	131
110	131	146	157	164	169
136	164	186	201	212	219
163	201	230	252	267	278
199	248	288	318	340	355
235	300	352	393	423	445
282	364	434	488	530	560
331	436	525	598	653	695
391	522	638	732	807	863
454	618	764	887	984	1060
532	722	910	1026	1204	1302
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]54	131970
13702	28652	50774	79403	112804	148847
148[illegible]0	31275	55974	88252	126299	167672
15944	34082	61575	97922	141136	188556
17180	37108	67696	108527	157564	211782
8467	40340	74280	120092	175586	237489
9858	43819	81457	132751	195491	266006
1301	47527	89162	146520	217280	2974[illegible]5
2856	51508	97539	161554	241279	332337
4473	55748	106522	177884	267507	370733
6207	60289	116263	195666	296320	413112
8009	65117	126692	214944	327748	459718
69941	70281	137977	235899	362198	511045
61943	75762	150042	258569	399705	567377
64085	81612	163069	283161	440725	629281
6308	87816	176978	309729	465315	697[illegible]

Mm 2 *b.*

sive addendo sive subtrahendo : num vero omnes Potestates prodeant, singulæque semel, sic exploro. Sit

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^1 + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.,$$

manifestum vero est, si x^3 loco x scribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^1} = bx^{-6} + ax^{-3} + 1 + \alpha x^3 + \beta x^6 + \gamma x^9 + \&c.:$$

Hinc igitur reperitur $P = \&c.$

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 + \beta x^5 + \beta x^6 + \beta x^7 + \&c.,$$

quæ expressio cum assumta comparata dabit

$$\alpha = 1;\ \beta = \alpha;\ \gamma = \alpha;\ \delta = \alpha;\ \varepsilon = \beta;\ \zeta = \beta;\ \&c., \ \& \ a = 1,\ b = a,\ c = a,\ d = a,\ e = b,\ \&c..$$

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c. + x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c.,$$

unde patet omnes ipsius x Potestates, tam affirmativas quam negativas, hic occurrere, atque adeo omnes numeros ex terminis progressionis geometricæ triplæ, vel addendo vel subtrahendo, formari posse; & unumquemque numerum unico tantum modo.

CAPUT XVII.

De usu Serierum recurrentium in radicibus æquationum indagandis.

332. Indicavit *Vir Celeb. Daniel* Bernoulli insignem usum Serierum recurrentium in investigandis radicibus æquationum cujusvis gradus, in *Coment. Acad. Petropol. Tomo III*, ubi ostendit, quemadmodum cujusque æquationis algebraicæ, quotcunque fuerit dimentionum, valores radicum veris proximi ope Serierum recurrentium assignari queant. Quæ inventio, cum sæpe numero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim præter expectationem evenit, ut nulla æquationis radix ope hujus methodi cognosci queat. Quocirca, ut vis hujus methodi clarius perspiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur sequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.;$$

cujus coëfficientes A, B, C, D, &c., ita determinantur ut sit

$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + \beta A + c$$
$$D = \alpha C + \beta B + \gamma A + d$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
&c.

Terminus autem generalis, seu coëfficiens Potestatis z^n, invenitur ex resolutione fractionis propositæ in fractiones simplices, quarum denominatores sint Factores denominatoris $1 - \alpha z - \beta zz - \gamma z^3 -$ &c., uti (*Cap. XIII.*) est ostensum.

334. Forma autem termini generalis potissimum pendet ab indole Factorum simplicium denominatoris, utrum sint reales an imaginarii, & utrum sint inter se inæquales & eorum bini pluresve æquales. Quos varios casus ut ordine percurramus, ponamus primum omnes denominatoris Factores simplices cum reales esse tum inter se inæquales. Sint ergo Factores simplices denominatoris omnes $(1-pz)(1-qz)(1-rz)(1-sz)$ &c., ex quibus fractio proposita in sequentes fractiones simplices resolvatur $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz} + \frac{D}{1-sz} +$ &c.. Quibus cognitis erit Seriei recurrentis terminus generalis $= z^n\,(Ap^n + Bq^n + Cr^n + Ds^n +$ &c.$)$, quem statuamus $= Pz^n$; sit scilicet P coëfficiens Potestatis z^n, sequentiumque Q, R, &c., ita ut Series recurrens fiat

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{\&c.}$$

335. Ponamus jam n esse numerum maximum, seu Seriem recurrentem ad plurimos terminos esse continuatam; quoniam numerorum inæqualium Potestates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diversitas in Potestatibus

LIB. I. Ap^n, Bq^n, Cr^n, &c., ut ea, quæ oritur ex maximo numerorum p, q, r, &c., reliquas magnitudine longe ſuperet, præ eaque reliquæ penitus evaneſcant, ſi n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r, &c., ſint inter ſe inæquales, ponamus inter eos p eſſe maximum; ac propterea, ſi n ſit numerus infinitus, fiet $P = Ap^n$; ſin autem n ſit numerus vehementer magnus erit tantum proxime $P = Ap^n$. Simili vero modo erit $Q = Ap^{n+1}$, ideoque $\frac{Q}{P} \doteq p$. Unde patet, ſi Series recurrens jam longe fuerit producta, coëfficientem cujuſque termini per præcedentem diviſum proxime eſſe exhibiturum valorem maximæ litteræ p.

336. Si igitur in fractione propoſita

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

denominator habeat omnes Factores ſimplices reales & inter ſe inæquales, ex Serie recurrente inde orta cognoſci poterit unus Factor ſimplex, is ſcilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coëfficientes numeratoris a, b, c, d, &c., in computum ingrediuntur, ſed quicunque ii ſtatuantur, tamen denique idem verus valor litteræ maximæ p invenitur. Verus quidem valor ipſius p tum demum innoteſcit, quando Series in infinitum fuerit continuata; interim tamen ſi jam plures ejus termini fuerint formati, eo propius valor ipſius p cognoſcetur, quo major fuerit terminorum numerus, & quo magis littera iſta p excedat reliquas q, r, s, &c.: perinde vero eſt utrum hæc maxima littera p fuerit ſigno + an ſigno — affecta, quoniam ejus Poteſtates æque increſcunt.

337. Quemadmodum nunc hæc inveſtigatio ad inventionem radicum æquationis cujuſvis algebraïcæ accommodari

possit, satis est perspicuum. Ex Factoribus enim denominatoris $1 - \alpha z - \beta z z - \gamma z^3 - \delta z^4 -$ &c., cognitis facile assignantur radices æquationis hujus CAP. XVII.

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, si Factor fuerit $1 - pz$, hujus æquationis radix una futura sit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix æquationis $1 - \alpha z - \beta z^2 - \gamma z^3 -$ &c. $= 0$. Vel, si ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejusdem methodi ope eruitur maxima hujus æquationis radix $x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter se inæquales, harum radicum maxima sequenti modo reperietur. Formetur ex coëfficientibus hujus æquationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, assumendo pro arbitrio numeratorem, seu, quod eodem redit, assumendo pro libitu terminos initiales; quæ sit

$$A + Bz + Cz^2 + Dz^3 + \dots\dots\dots\dots + Pz^n + Qz^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximæ x pro æquatione proposita, eo propius, quo major fuerit numerus n.

EXEMPLUM I.

Sit propofita ifta æquatio $xx - 3x - 1 = 0$, *cujus maximam radicem inveniri oporteat.*

Formetur fractio $\frac{a + bz}{1 - 3z - zz}$, unde pofitis duobus primis terminis 1, 2, orietur ifta Series recurrens

1, 2, 7, 23, 76, 251, 829, 2738, &c.,

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propofitæ maximæ. Valor autem hujus fractionis in partibus decimalibus expreffus eft

3, 3027744

æquationis vero radix maxima eft $= \frac{3 + \sqrt{13}}{2} =$

3, 3027756,

quæ inventam fuperat tantum una parte millionefima. Ceterum notandum eft fractiones $\frac{Q}{P}$ alternatim vera radice effe majores & minores.

EXEMPLUM II.

Propofita fit ifta æquatio $3x - 4x^3 = \frac{1}{2}$ *cujus radices exhibent Sinus trium Arcuum, quorum triplorum Sinus eft* $= \frac{1}{2}$.

Æquatione perducta ad hanc formam $0 = 1 - 6x * + 8x^3$, quæratur hujus, ut in numeris integris maneamus, radix minima, ita ut non opus fit pro x ponere $\frac{1}{2}$. Formetur ergo hæc fractio

$$\frac{a + bx + cxx}{1 - 6x * + 8x^3}$$

ex qua fumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia

quia hoc modo calculus facillime expeditur, orietur hæc Series recurrens, omittendis poteſtatibus ipſius x quia tantum coëfficientibus opus eſt,

$$0;\ 0;\ 1;\ 6;\ 36;\ 208;\ 1200;\ 6912;\ 39808;\ 229248.$$

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} = \frac{311}{1791} = 0,1736515$, quæ propterea eſſe deberet Sinus anguli 10°; hic autem ex tabulis eſt $0,1736482$, quem ſuperat radix inventa parte $\frac{33}{10000000}$. Facilius autem hæc eadem radix inveniri poteſt ponendo $x = \frac{1}{2}y$, ut prodeat æquatio $1 - 3y * + y^3 = 0$, ex qua ſimili modo tractata oritur Series

$$0,\ 0,\ 1,\ 3,\ 9,\ 26,\ 75,\ 216,\ 622,\ 1791,\ 5157\ \&c.,$$

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} = \frac{199}{573} = 0,3472949$, unde fit $x = \frac{1}{2}y = 0,1736479$, qui valor decies propius accedit quam præcedens.

EXEMPLUM III.

Si deſideretur ejuſdem æquationis propoſitæ $0 = 1 - 6x * + 8x^3$, *radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 * - 3y + 1 = 0$. Cujus æquationis radix maxima reperietur per Seriem recurrentem cujus ſcala relationis eſt $0,\ 3 - 1$, unde ergo oritur, ſumtis tribus terminis initialibus pro arbitrio,

$$1,\ 1,\ 1,\ 2,\ 2,\ 5,\ 4,\ 13,\ 7,\ 35,\ 8,\ 98 - 11,\ \&c.,$$

in qua Serie cum ad terminos negativos perveniatur, id indicio eſt maximam radicem eſſe negativam, eſt enim $x = -$

 fin. $70^\circ = - 0,9396926$. Quare hujus ratio in terminis initialibus est habenda, hoc modo

$$1 - 2 + 4 - 7 + 14 - 25 + 49 - 89 + 172 - 316 + 605 - \&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = - 0,957$, quæ a veritate vehementer abludit.

339. Ratio hujus dissensus potissimum est, quod æquationis propositæ radices sint *fin.* 10°, *fin.* 50°, & $-$*fin.* 70°, quarum binæ maximæ tam parum a se invicem discrepant, ut in Potestatibus, ad quas Seriem continuavimus, secunda radix *fin.* 50° adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evanescant. Hincque etiam saltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi. Sic, sumendo

$$y = \frac{-316}{172}, \text{ fit } x = \frac{-158}{172} = \frac{-79}{86} = - 0,918.$$

Nam, quoniam Potestates radicis maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Potestates secundæ radicis adduntur & tolluntur: quamobrem, quo hæc discrepantia fiat insensibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri potest, transmutando æquationem ope idoneæ substitutionis in aliam formam, cujus radices sibi non amplius sint tam vicinæ. Sic, si in æquatione $0 = 1 - 6x + 8x^3$ cujus radices sunt $-$ *fin.* 70°, $+$ *fin.* 50°, $+$ *fin.* 10°, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 -$ *fin.* 70°, $1 +$ *fin.* 50°; $1 +$ *fin.* 10°; ideoque ejus radix minima erit $1 -$ *fin.* 70°, cum tamen hæc *fin.* 70° esset radix maxima æquationis præcedentis; atque $1 +$ *fin.* 50° nunc est radix maxima, cum *fin.* 50° ante esset media. Atque hoc modo quævis radix per substitutionem in maximam minimamve radicem novæ æqua-

tionis transmutari, ideoque per methodum hic traditam inveniri poterit. Quia praeterea in hoc exemplo radix $1 - \mathit{fin}. 70^\circ$ multo minor est, quam binae reliquae, etiam facile per Seriem recurrentem proxime cognoscetur. CAP. XVII.

EXEMPLUM IV.

Invenire radicem minimam aequationis $0 = 8y^3 - 24yy + 18y - 1$, *quae ab unitate subtracta relinquet Sinum anguli* 70°.

Ponatur $y = \frac{1}{2} z$, ut fit $0 = z^3 - 6zz + 9z - 1$, cujus radix minima invenietur per Seriem recurrentem, cujus scala relationis est $9, -6, +1$, pro radice autem maxima invenienda scala relationis sumi deberet $6, -9, +1$. Pro minima ergo formetur haec Series

$1, 1, 1, 4, 31, 256, 2122, 17593; 145861;$ &c.,

erit ergo proxime $z = \frac{17593}{145861} = 0, 12061483$ & $y = 0, 06030741$, atque $\mathit{fin}. 70^\circ = 1 - y = 0, 93969258$, quae a veritate ne in ultima quidem figura discrepat. Ex hoc ergo exemplo intelligitur quantam utilitatem idonea transformatio aequationis ope substitutionis ad inventionem radicum afferat, & quod hoc pacto methodus tradita non solum ad maximas minimasve radices adstringatur, sed etiam omnes radices exhibere queat.

341. Cognita ergo jam quacunque aequationis propositae radice proxime, ita ut, verbi gratia, numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ seu $x = y + k$, hocque modo prodibit aequatio, cujus radix minima erit $= x - k$, quae igitur si per Series recurrentes indagetur, quod facillime fiet, quia haec radix multo minor erit, quam ceterae, si ea ad k addatur habebitur radix vera ipsius x, pro aequatione proposita. Hoc vero artificium tam late patet, ut etiamsi aequatio contineat radices imaginarias, usum suum retineat.

342. Imprimis autem fine hoc artificio radix cognofci nequit, cui datur alia æqualis fed figno contrario affecta. Scilicet, fi æquatio cujus maxima radix p, eamdem radicem habeat $-p$, tum, etiamfi Series recurrens in infinitum continuetur, tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo illuftremus, propofita æquatio $x^3 - x^2 - 5x + 5 = 0$, cujus maxima radix eft $\sqrt{5}$, præter quam vero ineft quoque $-\sqrt{5}$. Si igitur modo ante præfcriptio, pro radice maxima invenienda, utamur, atque Seriem recurrentem formemus ex fcala relationis 1, $+5$, -5, quæ erit

$$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \&c.,$$

ubi ad nullam rationem conftantem pervenitur. Termini vero alterni rationem æquabilem induunt, quorum fi quifque per præcedentem dividatur, reperietur quadratum maximæ radicis, fic enim eft proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo termini tantum alterni fefe ad rationem conftantem componunt, toties quadratum radicis quæfitæ proxime obtinetur. Ipfa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit $1 - 3y - 5yy - y^3 = 0$, cujus radix minima cognofcetur ex Serie

$$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \&c.,$$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ eft proxime $= \sqrt{5}$, quæ eft radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recurrens formatur, a noftro arbitrio pendet, tamen idonea ejus conftitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim affumtis, ut fupra, Factoribus denominatoris (334), fit terminus generalis Seriei recurrentis $= z^n (Ap^n + Bq^n + Cr^n + \&c.)$, ifti coëfficientes A, B, C, &c., per numeratorem fractionis determinantur; unde fieri poteft, ut A five magnum five parvum valorem obtineat: priori cafu radix maxima p cito reperitur, pofteriore vero tarde.

Quin etiam numerator ita accipi poteſt ut A prorſus evaneſcat, quo caſu, etiamſi Series in infinitum continuetur, tamen nunquam radicem maximam p præbebit. Hoc autem evenit ſi numerator ita accipiatur, ut ipſe eundem habeat Factorem $1 - pz$, ſic enim ex computo penitus tolletur. Sic, ſi proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix eſt $= 3$, indeque formetur fractio

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis ſit ſcala relationis $6, -10, +3$

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorſus non convergunt ad rationem, $1 : 3$. Eadem enim Series oritur ex fractione $\frac{1}{1 - 3z + zz}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita aſſumi poteſt, ut per Seriem recurrentem quævis radix æquationis reperiatur, quod fiet ſi numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui reſpondet radix quam velimus. Sic, ſi in priori exemplo ſumatur numerator $1 - 3z + zz$, fractio $\frac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$, dabit hanc Seriem recurrentem $1, 3, 9, 27, 81, 243$, &c., quæ, cum ſit geometrica, ſtatim monſtrat radicem $x = 3$. Fractio enim illa æqualis eſt huic ſimplici $\frac{1}{1 - 3z}$. Hinc apparet, ſi termini initiales, quos pro lubitu aſſumere licet, ita accipiantur, ut progreſſionem geometricam conſtituant, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipſam radicem eſſe exhibituram, etiamſi neque ſit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, hujuſmodi numerator debet eligi, qui

LIB. I. cum denominatore nullum Factorem habeat communem, quod fiet si pro numeratore unitas accipiatur, unde terminus primus Seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix æquationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic, proposita æquatione $y^3 * - 3y + 1 = 0$, cujus radix maxima desideratur, ex scala relationis $0, + 3, - 1$ incipiendo ab unitate sequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297, \\ - 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime $y = \frac{-6544}{3517} = -1,860676$, quæ esse debebat $= -1,86793852$. Ratio autem supra est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima, simulque sit affirmativa.

346. His probe perpensis, quæ cum in genere tum ad exempla allata monuimus, summa utilitas hujus methodi ad investigandas æquationum radices luculenter perspicietur; artificia vero, quibus operatio contrahi, eoque promtior reddi queat, satis quoque sunt indicata; ita ut nihil insuper addendum esset, nisi casus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus existentibus $1 - qz$, $1 - rz$, &c.. Seriei ergo recurrentis hinc natæ terminus generalis erit $= z^n((n+1)Ap^n + Bp^n + Cq^n + \&c.)$, quæ cujusmodi valorem sit adeptura, si n fuerit numerus

vehementer magnus, duo casus sunt distinguendi, alter quo p est numerus major reliquis q, r, &c., alter quo p non præbet radicem maximam. Casu priori, quo p simul est radix maxima, ob coëfficientem $(n+1)$ reliqui termini $Bp^n + Cq^n$ &c., non tam cito præ eo evanescent, quam ante: sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)Ap^n$ præ Bq^n evanescet, ideoque investigatio radicis maximæ admodum evadet molesta. CAP. XVII.

EXEMPLUM I.

Sit proposita æquatio $x^3 - 3xx + 4 = 0$, *cujus maxima radix* 2 *bis occurrit.*

Quæratur ergo maxima radix hæc modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z * + 4z^3}$$

quæ dabit hanc Seriem recurrentem

$$1,\ 3,\ 9,\ 23,\ 57,\ 135,\ 313,\ 711,\ 1593,\ \&c.,$$

ubi quidem quivis terminus per præcedentem divisus dat quotum binario majorem. Cujus ratio ex termino generali facillime patet, rejectis enim in eo terminis Bp^n, Cq^n &c., erit terminus potestati z^n respondens $= (n+1)Ap^n + Bp^n$, sequens $= (n+2)Ap^{n+1} + Bp^{n+1}$, qui per illum divisus dat $\frac{(n+2)A+B}{(n+1)A+B}p > p$, nisi n jam in infinitum excreverit.

EXEMPLUM II.

Sit jam proposita æquatio $x^3 - xx - 5x - 3 = 0$, *cujus maxima radix* $= 3$, *reliquæ duæ æquales* $= -1$, & quæratur maxima radix ope Seriei recurrentis, cujus scala relationis est $1, +5, +3$; unde oritur

$$1, 1, 6, 14, 47, 135, 412, 1228, \&c.,$$

quæ ideo satis cito valorem 3 exhibet, quod Potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox præ Potestatibus ipsius 3 evanescant.

EXEMPLUM III.

Sin antem proponeretur æquatio $x^3 + xx - 8x - 12 = 0$, cujus radices sunt $3, -2, -2$, multo tardius maxima sese prodet. Orietur enim hæc Series

$$1, -1, 9, -5, 65, 3, 457, 347, 3345, 4915, \&c.,$$

quæ adhuc longissime continuari deberet, antequam pateret, radicem inde oriundam esse $= 3$.

347. Simili modo si tres Factores essent æquales, ita ut denominatoris Factor unus esset $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $= z^n\left(\frac{(n+1)(n+2)}{1.\ 2}Ap^n + (n+1)Bp^n + Cp^n + Dq^n + Cr^n \&c.\right)$ Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Potestates q^n, r^n &c. præ p^n evanescant, tum ex Serie recurrente orietur radix $=$

$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C}p,$$

quæ, nisi sit n numerus maximus & quasi infinitus, verum ipsius

ipsius p valorem indicabit. Erit autem iste radice valor $= p +$

$$\frac{(n+2)A+B}{\frac{1}{2}(n+1)(n+2)A+(n+1)B+C}p.$$

Quod si autem p non fuerit radix maxima, tum inventio maxime multo magis adhuc impedietur; unde sequitur æquationes, quæ contineant radices æquales, hac methodo per Series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a+bz+cz^2+dz^3+\&c.}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\&c.}$$

Factores denominatoris reales $1-qz$, $1-rz$, &c., insuperque Factor trinomialis $1-2pz.\,cos.\,\varphi+ppzz$ continens duos Factores simplices imaginarios. Quod si ergo Series recurrens ex illa fractione orta fuerit

$A+Bz+Cz^2+Dz^3+\ldots\ldots\ldots\ldots+Pz^n+Qz^{n+1}$,

erit, per ea quæ supra exposuimus, coëfficiens $P =$ $\frac{A.\,fin.\,(n+1)\varphi+B.\,fin.\,n\varphi}{fin.\,\varphi}p^n+Cq^n+Dr^n+\&c.$. Si igitur numerus p minor fuerit, quam unus ceterorum q, r, &c., ita ut maxima radix æquationis

$$x^m-\alpha x^{m-1}-\beta x^{m-2}-\gamma x^{m-3}-\&c.=0;$$

sit realis, tum ea per Series recurrentes æque reperietur, ac si nullæ radices inessent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imaginarias non perturbabitur, si hæ ita fuerint comparatæ, ut binarum, quæ Factorem realem componunt, productum non sit

LIB. I. majus quadrato radicis maximæ. Sin autem binæ ejusmodi insint radices imaginariæ, ut earum productum adæquet vel adeo superet quadratum maximæ radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod Potestas p^n, præ simili Potestate radicis maximæ nunquam evanescit, etiamsi Series in infinitum continuetur. Cujus exempla illustrationis causa hic adjicere visum est.

EXEMPLUM I.

Sit proposita æquatio $x^3 - 2x - 4 = 0$, *cujus radicem maximam investigari oporteat.*

Resolvitur hæc æquatio in duos Factores $(x-2)(xx+2x+2)$; unde unam habet radicem realem 2 & duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quam ob rem ea per modum hactenus traditum cognosci poterit. Formetur ergo Series recurrens ex scala relationis $0, +2, +4$, quæ erit

1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, &c.,

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM II.

Proposita sit æquatio $x^3 - 4xx + 8x - 8 = 0$, *cujus radix una realis est* 2, *binarum imaginariarum productum vero* $= 4$, *ideoque æquale quadrato radicis realis* 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 - 2yy + 2y - 1 = 0$, unde formetur Series recurrens

1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, &c.,

in qua cum iidem termini perpetuo revertantur, nihil inde

aliud colligi poteſt, niſi radicem maximam vel non eſſe realem, vel dari imaginarias, quarum productum æquale ſit aut ſuperet quadratum radicis realis. CAP. XVII.

EXEMPLUM III.

Sit jam propoſita æquatio $x^3 - 3xx + 4x - 2 = 0$, *cujus radix realis eſt* 1, *imaginariarum vero productum* $= 2$.

Formetur ergo ex ſcala relationis $3, -4, +2$, Series $1, 3, 5, 5; 1, -7, -15, -15; +1, 33, 65, 65, 1$, &c., in qua cum termini modo fiant affirmativi, modo negativi, radix realis 1 inde nullo modo cognoſci poterit. Hujuſmodi vero revolutiones ſemper oſtendunt radicem, quam Series præbere debebat, eſſe imaginariam; hic enim radices imaginariæ poteſtate ſunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp majus quam ullius radicis realis quadratum, ita ut præ p^n reliquæ poteſtates q^n, r^n, &c., evaneſcant ſi n ſit numerus infinitus. Hoc ergo caſu fiet $P = \frac{A.\,\text{ſin.}(n+1)\varphi + B.\,\text{ſin.}\,n\varphi}{\text{ſin.}\,\varphi}\,p^n$, & $Q = \frac{A.\,\text{ſin.}(n+2)\varphi + B.\,\text{ſin.}(n+1\varphi)}{\text{ſin.}\,\varphi}\,p^{n+1}$ ideoque $\frac{Q}{P} = \frac{A.\,\text{ſin.}(n+2)\varphi + B.\,\text{ſin.}(n+1)\varphi}{A.\,\text{ſin.}(n+1)\varphi + B.\,\text{ſin.}\,n\varphi}\,p$. Quæ expreſſio nunquam valorem conſtantem induet, etiamſi n ſit numerus infinitus. Sinus enim Angulorum perpetuo maxime manent mutabiles, ita ut mox ſint affirmativi mox negativi.

351. Interim tamen ſi fractiones ſequentes $\frac{R}{Q}$, $\frac{S}{R}$ ſimili modo ſumantur, indeque litteræ A & B eliminentur, ſimul numerus n ex calculo egredietur; reperietur enim $Ppp + R = 2Qp.\,\text{coſ.}\,\varphi$, unde fit $\text{coſ.}\,\varphi = \frac{Ppp + R}{2Qp}$; ſimiliter vero erit

LIB. I. $\cos.\,\varphi = \frac{Qpp + S}{2Rp}$, ex quorum duorum valorum comparatione fit $p = \sqrt{\frac{RR - QS}{QQ - PR}}$, atque $\cos.\,\varphi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(R^2 - QS)}}$. Quam ob rem si Series recurrens jam eo usque fuerit continuata, ut præ p^n reliquarum radicum Potestates evanescant, tum hoc modo Factor trinomialis $1 - 2pz.\cos.\,\varphi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur $\mathrm{A}Pp.\,\mathit{fin}.\,(n+2)\varphi + \mathrm{B}Pp.\,\mathit{fin}.\,(n+1)\varphi = \mathrm{A}Q.\,\mathit{fin}.\,(n+1)\varphi + \mathrm{B}Q.\,\mathit{fin}.\,n\varphi$, unde fit $\frac{\mathrm{A}}{\mathrm{B}} = \frac{Q.\,\mathit{fin}.\,n\varphi - Pp.\,\mathit{fin}.\,(n+1)\varphi}{Pp.\,\mathit{fin}.\,(n+2)\varphi - Q.\,\mathit{fin}.\,(n+1)\varphi}$. Pari ratione erit $\frac{\mathrm{A}}{\mathrm{B}} = \frac{R.\,\mathit{fin}.\,(n+1)\varphi - Qp.\,\mathit{fin}.\,(n+2)\varphi}{Qp.\,\mathit{fin}.\,(n+3)\varphi - R.\,\mathit{fin}.\,(n+2)\varphi}$; æquatis his duobus valoribus fiet

$0 = QQp.\,\mathit{fin}.\,n\varphi.\,\mathit{fin}.\,(n+3)\varphi - QR.\,\mathit{fin}.\,n\varphi.\,\mathit{fin}.\,(n+2)\varphi - PQpp.\,\mathit{fin}.(n+1)\varphi.\,\mathit{fin}.(n+3)\varphi - QQp.\,\mathit{fin}.(n+1)\varphi.\,\mathit{fin}.(n+2)\varphi + QR.\,\mathit{fin}.(n+1)\varphi.\,\mathit{fin}.(n+1)\varphi + PQpp.\,\mathit{fin}.(n+1)\varphi.\,\mathit{fin}.(n+2)\varphi.$

Cum autem sit $\mathit{fin}.\,a.\,\mathit{fin}.\,b = \frac{1}{2}.\cos.(a-b) - \frac{1}{2}.\cos.(a+b)$ fiet $0 = \frac{1}{2}QQp.(\cos.3\varphi - \cos.\varphi) + \frac{1}{2}QR.(1 - \cos.2\varphi) + \frac{1}{2}PQpp.(1 - \cos.2\varphi)$ quæ per $\frac{1}{2}Q$ divisa dat $(Ppp + R)(1 - \cos.2\varphi) = Qp.(\cos.\varphi - \cos.3\varphi)$. At est $\cos.\varphi = \cos.2\varphi.\cos.\varphi + \mathit{fin}.2\varphi.\,\mathit{fin}.\varphi$ & $\cos.3\varphi = \cos.2\varphi\cos.\varphi - \mathit{fin}.2\varphi.\,\mathit{fin}.\varphi$ unde $\cos.\varphi - \cos.3\varphi = 2\,\mathit{fin}.2\varphi.\,\mathit{fin}.\varphi = 4\,\mathit{fin}.\varphi^2 \times \cos.\varphi$ & $1 - \cos.\varphi = 2\,\mathit{fin}.\varphi^2$, ex quo erit $Ppp + R = 2Qp.\cos.\varphi$, & $\cos.\varphi = \frac{Ppp + R}{2Qp}$, atque $\cos.\varphi = \frac{Qpp + S}{2Rp}$: unde

superiores valores prodeunt, scilicet $p = \sqrt{\frac{RR - QS}{QQ - PS}}$ &
$cos. \varphi = \frac{QR - PS}{2\sqrt{(Q^3 - PR)(RR - QS)}}$.

353. Si denominator fractionis; ex qua Series recurrens formatur, plures habeat Factores trinomiales inter se æquales, tum, spectata forma termini generalis supra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione semper valor ejusdem radicis multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+ y$, atque novæ æquationis quæratur minima radix pro y, quæ addita ad illum valorem præbebit verum ipsius x valorem.

Exemplum.

Sit proposita ista æquatio $x^3 - 3xx + 5x - 4 = 0$, *cujus unam radicem fere esse* $= 1$ *inde constat, quod, posito* $x = 1$, *prodit* $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus scala relationis $2, 0, + 1$, quæ erit

$1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296$, &c.,

unde radix minima ipsius y erit proxime $\frac{1041}{2296} = 0,453397$, ita ut sit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod si autem Series quæcunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cujusnam æquationis radix sit futura quotus qui ex divisione unius termini per præcedentem oritur. Sint

$$P,\ Q,\ R,\ S,\ T,\ \&c.,$$

termini Seriei recurrentis a principio jam longissime remoti, ita ut cum progressione geometrica confundantur; sitque $T = \alpha S + \beta R + \gamma Q + \delta P$, seu scala relationis $\alpha, +\beta, +\gamma, +\delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit $\frac{R}{P} = xx$; $\frac{S}{P} = x^3$ & $\frac{T}{P} = x^4$, qui in superiori æquatione substituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

unde patet quotum $\frac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat, præterea vero docet fractionem $\frac{Q}{P}$ dare maximam æquationis radicem.

355. Potest quoque hæc methodus investigandarum radicum sæpenumero utiliter adhiberi, si æquatio sit infinita. Ad quod ostendendum proposita sit æquatio $\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \&c.$, cujus radix minima z exhibet Arcum 30°, seu Semiperipheriæ Circuli sextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \&c. = 0.$$

Hinc ergo formetur Series recurrens, cujus scala relationis est infinita, scilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0\ \&c.,$$

eritque Series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45},\ \&c.,$$

erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1631.3}{2408.4} = \frac{5043}{9632} = 0,52356$: CAP. XVII.
At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. Quoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmeticam & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cujus denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

LIB. I.

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\&c.}}}}}\quad \text{vel}\quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\&c.}}}}};$$

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Exposita ergo fractionum harum continuarum forma, primum videndum est, quemadmodum earum significatio consueto more expressa inveniri queat. Quæ ut facilius inveniri possit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in secunda, post in tertia & ita porro fractione; quo facto patebit fore

$$a = a$$

$$a+\frac{1}{b} = \frac{ab+1}{b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

&c..

359. Etsi in his fractionibus ordinariis non facile lex, secundum quam numerator ac denominator ex litteris a, b, c, d, &c., componantur, perspicitur, tamen attendenti statim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator est aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

meratore penultimo simplici : eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris a, b, c, d, &c., ex iis fractiones inventæ facile formabuntur hoc modo CAP. XVIII.

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+1}{b}; & \frac{abc+a+c}{bc+1}; & \frac{abcd+ab+ad+cd+1}{bcd+b+d} \end{array}$$

ubi quilibet numerator invenitur, si præcedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur ; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ usque ad eam litteram, quæ antecedenti imminet, inclusive continuata.

360. Simili modo altera fractionum continuarum forma

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\epsilon}{f + \&c.}}}}},$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$a = a$$

$$a + \frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c}} = \frac{abc + \beta a + \alpha c}{bc + \beta}$$

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d}}} = \frac{abcd + \beta ad + \alpha cd + \gamma ab + \alpha\gamma}{bcd + \beta d + \gamma b}$$

&c.,

quarum fractionum quæque ex binis præcedentibus sequentem in modum invenietur.

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+\alpha}{b}; & \frac{abc+\beta a+\alpha c}{bc+\beta}; & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}. \\ \alpha & \beta & \gamma & \delta & \epsilon \end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d, &c., infra autem subscribantur indices α, β, γ, δ, &c.. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$, tum sequentium quævis formabitur si antecedentium ultimæ numerator per indicem supra scriptum, penultimæ vero numerator per indicem infra scriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis sequentis: simili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum; & ex penultimo denominatore per indicem infra scriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatæ inclusive.

362. Quod si ergo hæ fractiones eousque continuentur quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuæ

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{a+\cfrac{\epsilon}{e+\text{\&c.}}}}}\quad\text{esse} = x,$$

atque manifestum est fractionem primam $\frac{1}{0}$ esse majorem

quam x; fecunda vero $\frac{a}{1}$ minor erit quam x; tertia $a + \frac{\alpha}{b}$ iterum vero valore erit major; quarta denuo minor, atque ita porro hæ fractiones alternatim erunt majores & minores quam x. Porro autem perfpicuum eft quamlibet fractionem propius accedere ad verum valorem x quam ulla præcedentium; unde hoc pacto citiffime & commodiffime valor ipfius x proxime obtinetur; etiamfi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ, &c., non nimis crefcant; fin autem omnes ifti numeratores fuerint unitates, tum appropinquatio nulli incommodo eft obnoxia. CAP. XVIII.

363. Quo ratio hujus appropinquationis ad verum fractionis continuæ valorem melius percipiatur, confideremus fractionum inventarum differentias. Ac, prima quidem $\frac{1}{0}$ prætermiffa, differentia inter fecundam ac tertiam eft $= \frac{\alpha}{b}$; quarta a tertia fubtracta relinquit $\frac{\alpha\beta}{b(bc+\beta)}$; quarta a quinta fubtracta relinquit $\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma)}$, &c.. Hinc exprimetur valor fractionis continuæ per Seriem terminorum confuetam hoc modo, ut fit

$$x = a + \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in Seriem terminorum, quorum figna alternantur, convertendi, fi quidem prima littera a evanefcat. Si enim fuerit

$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.}}}}},$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \&c..$$

Unde, si $\alpha, \beta, \gamma, \delta$, &c. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe consideratis, poterit vicissim Series quæcunque terminorum alternantium in fractionem continuam converti, seu fractio continua inveniri cujus valor æqualis sit summæ Seriei propositæ. Sit enim proposita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, singulis terminis cum Serie ex fractione continua orta comparandis.

$A = \frac{\alpha}{b}$; hincque $\alpha = Ab$,

$\frac{B}{A} = \frac{\beta}{bc+\beta}$; unde fit $\beta = \frac{Bbc}{A-B}$

$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}$; $\gamma = \frac{Cd(bc+\beta)}{b(B-C)}$

$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}$; $\delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$.

&c..

At, cum sit $\beta = \frac{Bbc}{A-B}$, erit $bc + b = \frac{Abc}{A-B}$; unde

$\gamma = \frac{ACcd}{(A-B)(B-C)}$. Porro fit $bcd + \beta d + \gamma b =$ $(bc+\beta)d+\gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)}$, unde erit $\frac{bcd+\beta d+\gamma b}{bc+\beta} = \frac{Bd}{B-C}$ & $\delta = \frac{BDde}{(B-C)(C-D)}$; ſimili modo reperietur $\epsilon = \frac{CEef}{(C-D)(D-E)}$ & ita porro.

366. Quo iſta lex clarius appareat, ponamus eſſe

$$P = b$$
$$Q = bc + \beta$$
$$R = bcd + \beta d + \gamma d$$
$$S = bcde + \beta de + \gamma be + \delta bc + \beta\delta$$
$$T = bcdef + \&c.$$
$$V = bcdefg + \&c.,$$

erit ex lege harum expreſſionum

$$Q = Pc + \beta$$
$$R = Qd + \gamma P$$
$$S = Re + \delta Q$$
$$T = Sf + \epsilon R$$
$$V = Tg + \xi S$$
&c..

Cum igitur his adhibendis litteris ſit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\epsilon}{ST} - \&c.:$$

367. Quoniam ergo ponimus eſſe

$$x = A - B + C - D + E - F + \&c.,$$

erit

$$A = \frac{\alpha}{P};\ \alpha = AP.$$

$$\frac{B}{A} = \frac{\beta}{Q}\,;\quad \beta = \frac{BQ}{A}$$
$$\frac{C}{B} = \frac{\gamma P}{R}\,;\quad \gamma = \frac{CR}{BP}$$
$$\frac{D}{C} = \frac{\delta Q}{S}\,;\quad \delta = \frac{DS}{CQ}$$
$$\frac{E}{D} = \frac{\epsilon R}{T}\,;\quad \epsilon = \frac{ET}{DR}$$
&c. &c.

Porro vero differentiis sumendis habebitur

$$A - B = \frac{\alpha(Q - \beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$
$$B - C = \frac{\alpha\beta(R - \gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}$$
$$C - D = \frac{\alpha\beta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$
$$C - D = \frac{\alpha\beta\gamma\delta(T - \epsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T}\,,$$
&c. &c. &c.

Si bini igitur in se invicem ducantur, fiet

$$(A-B)(B-C) = ABcd.\frac{P}{R}\,;\ \&\ \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)}$$
$$(B-C)(C-D) = BCde.\frac{Q}{S}\,;\ \&\ \frac{S}{Q} = \frac{BCed}{(B-C)(C-D)}$$
$$(D-E)(D-E) = CDef.\frac{R}{T}\,;\ \&\ \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$
&c..

Unde, cum sit $P = b\,;\ Q = \frac{\alpha c}{A-B} = \frac{Abc}{A-B}$, erit

$$\alpha = Ab$$
$$\beta = \frac{Bbc}{A-B}$$
$$\gamma = \frac{ACcd}{(A-B)(B-C)}$$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

&c. .

368. Inventis ergo valoribus numeratorum $\alpha, \beta, \gamma, \delta$, &c.; denominatores b, c, d, e, &c., arbitrio noſtro relinquuntur: ita autem eos aſſumi convenit, ut, cum ipſi ſint numeri integri, tum valores integros pro $\alpha, \beta, \gamma, \delta$, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum ſint integri an fracti. Ponamus eſſe numeros integros, atque quæſito ſatisfiet ſtatuendo

$$\begin{array}{lcl} b = 1 & & \alpha = A \\ c = A - C & & \beta = B \\ d = B - C & \text{unde fit} & \gamma = AC \\ e = C - D & & \delta = BD \\ f = D - E & & \varepsilon = CE \\ \text{\&c.} & & \text{\&c.} \end{array}$$

Quocirca, ſi fuerit,

$$x = A - B + C - D + E - F + \text{\&c.},$$

idem ipſius x valor per fractionem continuam ita exprimi poterit, ut ſit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{\&c.}}}}}}$$

369. Sin autem omnes termini Seriei ſint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{\&c.},$$

habebuntur pro $\alpha, \beta, \gamma, \delta$, &c., ſequentes valores

$$\alpha = \frac{b}{A}\;;\; \mathfrak{C} = \frac{Abc}{B-A}\;;\; \gamma = \frac{B^{\prime}cd}{(B-A)(C-B)}\;,$$

$$\delta = \frac{C^{\prime}de}{(C-B)(D-C)}\;;\; \epsilon = \frac{D^{\prime}ef}{(D-C)(E-D)}\;,\ \&c.,$$

Ponatur ergo ut ſequitur

$$\begin{array}{ll} b = A\,; & \alpha = 1 \\ c = B - A\,; & \mathfrak{C} = AA \\ d = C - B\,; & \gamma = BB \\ e = D - C\,; & \delta = CC \end{array}$$

&c.,

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \&c.}}}}\,.$$

E X E M P L U M I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propoſitæ valor ſit $= l\,2$, erit

$$l\,2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \&c.}}}}}}\,.$$

E X E M P L U M II.

Transformetur hæc Series infinita

$\frac{\pi}{4}$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Substitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}}};$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}},$$

quæ est expressio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit proposita ista Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

LIB. I. $\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}$

EXEMPLUM IV.

Quoniam, supra §. 178, invenimus esse

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n-2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n-2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n-2m + \&c.}}}}}}$$

370. Si Series proposita per continuos Factores progrediatur, ut sit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt sequentes determinationes

$$\alpha = \frac{b}{A};\ \beta = \frac{bc}{B-1};\ \gamma = \frac{Bcd}{(B-1)(C-1)};$$
$$\delta = \frac{Cde}{(C-1)(D-1)};\ \epsilon = \frac{Def}{(D-1)(E-1)};\ \&c.,$$

fiat ergo ut sequitur,

$$b = A; \quad \alpha = 1$$
$$c = B - 1; \quad \beta = A$$
$$d = C - 1; \quad \gamma = B$$
$$e = D - 1; \quad \delta = C$$
$$f = E - 1; \quad \epsilon = D$$

&c.,

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \&c.}}}}}$$

Exemplum I.

Quoniam, posito e numero cujus Logarithmus est $= 1$, supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \&c.,$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \&c.,$$

hæc series in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.: quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}};$$

unde, asymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}$$

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis ſumitur, coſinum eſſe $= 1 - \frac{1}{2} + \frac{1}{2.\,12} - \frac{1}{2.\,12.\,30} + \frac{1}{2.\,12.\,30.\,56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Coſinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}};$$

ſeu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{20}{55 + \&c.}}}}$$

371. Sit Series inſuper cum geometrica conjuncta, ſcilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \&c.,$$

erit

$$\alpha = Ab\,;\ \beta = \frac{Abcz}{A - Bz}\,;\ \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)}\,;$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)}\,;\ \varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)}\,;\ \&c..$$

Ponatur nunc

$$\begin{array}{ll} b = 1\,; & \alpha = A \\ c = A - Bz\,; & \beta = Bz \\ d = B - Cz\,; & \gamma = ACz \\ e = C - Dz\,; & \delta = BDz, \end{array}$$

unde fiet

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \&c.}}}}$$

372. Quo autem hoc negotium generalius absolvamus, ponamus esse

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione instituta,

$$\alpha = \frac{Ab}{L};\ \beta = \frac{BLbcy}{AMz - BLy};\ \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)};$$

$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)};\ \&c.,$$

statuantur valores b, c, d, &c., sequenti modo

$$\begin{array}{lcl} b = L; & & \alpha = A \\ c = AMz - BLy; & & \beta = BLLy \\ d = BNz - CMy; & \text{erit} & \gamma = ACM^2yz \\ e = COz - DNy; & & \delta = BDN^2yz \\ f = DPz - EOy; & & \epsilon = CEO^2yz \\ \&c. & & \&c. \end{array}$$

unde Series proposita per sequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \&c.}}}}$$

373. Habeat denique Series proposita hujusmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque sequentes valores prodibunt

$$\alpha = \frac{Ab}{L};\ \beta = \frac{Bbcy}{Mz - By};\ \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)};$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)};\ \epsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)};$$

&c.,

ad valores ergo integros inveniendos fiat

$b = Lz;$	erit	$\alpha = Az$
$c = Mz - By;$		$\beta = BLyz$
$d = Nz - Cy;$		$\gamma = CMyz$
$e = Oz - Dy;$		$\delta = DNyz$
$f = Pz - Ey;$		$\epsilon = EOyz$
&c.		&c.

Unde valor Seriei propositæ ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}}.$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quan-

quam enim fractio continua transmutari potest in Seriem infinitam, cujus summa per methodos cognitas investigari queat, tamen plerumque istæ Series tantopere fiunt intricatæ, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit.

375. Quo autem clarius perspiciatur, dari ejusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujus omnes denominatores sunt inter se æquales; si enim hinc modo supra exposito, fractiones formemus

$$\begin{matrix} 0, & 2, & 2, & 2, & 2, & 2, & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, \ \&c.: \end{matrix}$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, si bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c..$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

LIB. II.

$$+ \frac{1}{4} - \frac{1}{2.\,2.\,5} + \frac{1}{2.\,5.\,12} - \frac{1}{2.\,12.\,29} + \&c.,$$

erit

$$x = \frac{1}{4} + \frac{1}{1.\,5} - \frac{1}{2.\,12} + \frac{1}{5.\,29} - \frac{1}{12.\,70}\ \&c.,$$

quæ Series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro hujusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt æquales, vel iidem revertuntur; ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit æqualis, facilis habetur modus earum summas explorandi. In exemplo enim proposito, cum sit

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

erit $x = \frac{1}{2+x}$, ideoque $xx + 2x = 1$ & $x + 1 = \sqrt{2}$; ita ut valor hujus fractionis continuæ sit $= \sqrt{2} - 1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Est enim $\sqrt{2} - 1$ tam prope $= \frac{29}{70}$, ut error sit insensibilis: namque, radicem extrahendo, erit

$$\sqrt{2} - 1 = 0,\ 41421356236,$$

atque

$$\frac{29}{70} = 0,\ 41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuæ commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem

$$x = \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \&c.}}}}};$$

erit $x = \frac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \sqrt{(1 + \frac{1}{4}aa)} = \frac{\sqrt{(aa+4)} - a}{2}$. Hæc ergo fractio continua inserviet valori radicis quadratæ ex numero $aa + 4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4, &c., reperientur $\sqrt{5}$; $\sqrt{2}$; $\sqrt{13}$; $\sqrt{5}$; $\sqrt{29}$; $\sqrt{10}$; $\sqrt{53}$; &c., perductis scilicet his radicibus ad formam simplicissimam: erit ergo

$$\begin{array}{l} 1,\ 1,\ 1,\ 1,\ 1,\ 1, \\ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \&c. = \frac{\sqrt{5} - 1}{2} \end{array}$$

$$\begin{array}{l} 2,\ 2,\ 2,\ 2,\ 2,\ 2 \\ \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \&c. = \sqrt{2} - 1 \end{array}$$

$$\begin{array}{l} 3,\ 3,\ 3,\ 3,\ 3,\ 3 \\ \frac{0}{1}, \frac{1}{3}, \frac{3}{10}, \frac{10}{33}, \frac{33}{109}, \frac{109}{360}, \&c. = \frac{\sqrt{13} - 3}{2} \end{array}$$

$$\begin{array}{l} 4,\ 4,\ 4,\ 4,\ 4,\ 4 \\ \frac{0}{1}, \frac{1}{4}, \frac{4}{17}, \frac{17}{72}, \frac{72}{305}, \frac{305}{1292}, \&c. = \sqrt{5} - 2 \end{array}$$

&c.

notandum autem eo promptiorem esse approximationem, quo major fuerit numerus a: sic in ultimo exemplo erit $\sqrt{5} = 2 \frac{305}{1292}$, ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

Lib. I. 378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \&c.}}}}}};$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b+x}} = \frac{b+x}{ab+1+ax}; \text{ ideoque } axx + abx = b,$$

&

$$x = -\tfrac{1}{2}b \pm \surd(\tfrac{1}{4}bb + \tfrac{b}{a}) = \frac{-ab + \surd(aabb + 4ab)}{2a}.$$ Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a = 2$, $b = 7$; erit $x = \frac{-14 + \surd 14.18}{4} = \frac{-7 + 3\surd 7}{2}$; at valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{matrix} & 2, & 7, & 2, & 7, & 2, & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, & \&c., \end{matrix}$$

Erit ergo proxime $\frac{-7 + 3\surd 7}{2} = \frac{239}{510}$ & $\surd 7 = \frac{2024}{765} = 2,6457516$; at revera est $\surd 7 = 2,64575131$; ita ut error minor sit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \&c.}}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c+x}}} = \cfrac{1}{a + \cfrac{c+x}{bx+bc+1}} = \frac{bx+bc+1}{(ab+1)x + abc + a + c};$$

unde $(ab+1)xx + (abc + a - b + c)x = bc + 1$ atque

$$x = \frac{-abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab+1)};$$ CAP. XVIII.

ubi quantitas poſt ſignum radicale poſita iterum eſt ſumma duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inſervit, niſi ad quos prima forma jam ſuffecerat. Simili modo ſi quatuor litteræ a, b, c, d, continuo repetitæ denominatores fractionis continuæ conſtituant, tum ea plus non inſerviet quam ſecunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, ſimul inſervient æquationibus quadratis reſolvendis; quod quidem ex ipſo calculo eſt manifeſtum, dum x per æquationem quadraticam affectam determinatur. Poteſt autem viciſſim facile cujuſque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit propoſita iſta æquatio

$$xx = ax + b,$$

ex qua, cum ſit $x = a + \frac{b}{x}$, ſubſtituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}},$$

ſimili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \&c.}}},$$

quæ autem, cum numeratores b non ſint unitates, non tam commode adhiberi poteſt.

381. Ut autem uſus in arithmetica oſtendatur, primum notandum eſt omnem fractionem ordinariam in fractionem con-

LIB. I. tinuam converti posse. Sit enim proposita fractio $x = \frac{A}{B}$, in qua sit $A > B$; dividatur A per B, sitque quotus $= a$ & residuum C; tum per hoc residuum C dividatur præcedens divisor B, prodeatque quotus b & relinquatur residuum D, per quod denuo præcedens divisor C dividatur; sicque hæc operatio, quæ vulgo ad maximum communem divisorem numerorum A & B investigandum usurpari solet, continuetur, donec ipsa finiatur; sequenti modo

$$\begin{array}{l} B\,)\,A\,(\,a \\ \quad C\,)\,B\,(\,b \\ \qquad D\,)\,C\,(\,c \\ \qquad\quad E\,)\,D\,(\,d \\ \qquad\qquad F \text{ \&c.}, \end{array}$$

eritque per naturam divisionis

$$\begin{array}{lll} A = aB + C; & \text{unde } \frac{A}{B} = a + \frac{C}{B}; & \\ B = bC + D; & \frac{B}{C} = b + \frac{D}{C}; & \frac{C}{B} = \frac{1}{b + \frac{D}{C}} \\ C = cD + E; & \frac{C}{D} = c + \frac{E}{D}; & \frac{D}{C} = \frac{1}{c + \frac{E}{D}} \\ D = dE + F; & \frac{D}{E} = d + \frac{F}{E}; & \frac{E}{D} = \frac{1}{d + \frac{F}{E}} \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

hinc, sequentes valores in præcedentibus substituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \frac{D}{C}} = a + \frac{1}{b + \frac{1}{c + \frac{E}{D}}},$$

unde tandem x per meros quotos inventos a, b, c, d, &c., sequentem in modum exprimetur, ut sit

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \&c.}}}}}$$

EXEMPLUM I.

Sit proposita ista fractio $\frac{1461}{59}$, quæ sequenti modo in fractionem continuam transmutabitur, cujus omnes numeratores erunt unitates. Instituatur scilicet eadem operatio, qua maximus communis divisor numerorum 59 & 1461 quæri solet.

```
59) 1461 (24
    118
    ---
    281
    236
    ---
     45) 59 (1
         45
         --
         14) 45 (3
             42
             --
              3) 14 (4
                 12
                 --
                  2) 3 (1
                     2
                     -
                     1) 2 (2
                        2
                        -
                        0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo transmutari poterunt, sit enim proposita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio instituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
&c.	209164	

Ex qua operatione jam patet omnes denominatores esse 2, atque adeo esse $\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\&c.}}}}}$, cujus expressionis ratio jam ex superioribus patet.

Exemplum III.

Imprimis vero etiam hic attentione dignus est numerus e; cujus logarithmus est $= 1$, qui est $e = 2,718281828459$, unde oritur $\frac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	&c.	

si iste calculus exactius adhuc, assumto valore ipsius e, ulterius continuetur, tum prodibunt isti quoti

1, 6, 10, 14, 18, 22, 26, 30, 34, &c.,

qui, demto primo, progressionem arithmeticam constituunt, unde patet fore

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\&c.}}}}}}},$$

cujus fractionis ratio ex calculo infinitesimali dari potest.

382. Cum igitur ex hujusmodi expressionibus fractiones erui queant, quæ quam citissime ad verum valorem expressionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita cujus numerator & denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt quæ, etiamsi propositæ non sint penitus æquales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quæruntur fractiones minoribus numeris expressæ, quæ tam prope exhauriant valorem fractionis cujuspiam in numeris majoribus propositæ, quantum fieri poterit numeris non majoribus. Fractiones autem nostra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus constantes dentur quæ propius accedant.

EXEMPLUM I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri majores adhibeantur. Si fractio decimalis cognita

$$3,1415926535 \text{ \&c.},$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3,\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ \&c.,$$

ex quibus sequentes fractiones formabuntur,

$$\frac{1}{0},\ \frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \&c.,$$

 ſecunda fractio jam oſtendit eſſe diametrum ad peripheriam ut 1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor ſit parte $\frac{1}{113 \cdot 33102}$. Ceterum hæ fractiones alternatim vero ſunt majores minoreſque.

EXEMPLUM II.

Exprimatur ratio diei ad annum ſolarem medium in numeris minimis proxime. Cum annus iſte ſit 365^{d}, 5^{h}, $48'$, $55''$, continebit in fractione annus unus $365 \frac{20935}{86400}$ dies. Tantum ergo opus eſt ut hæc fractio evolvatur, quæ dabit ſequentes quotos

$$4,\ 7,\ 1,\ 6,\ 1,\ 2,\ 2,\ 4$$

unde iſtæ eliciuntur fractiones

$$\frac{0}{1},\ \frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{55}{227},\ \frac{63}{260},\ \frac{181}{747},\ \&c..$$

Horæ ergo cum minutis primis & ſecundis, quæ ſupra 365 dies adſunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde ſequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inſerat 100 dies, *Gregorianus* quaternis ſeculis tres annos biſſextiles in communes convertit.

FINIS TOMI PRIMI.